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OPTIMIZATION OF THERMOELECTRIC COOLING

BY THE USE OF CASCADES

A THESIS

Presented to

The Faculty of the Graduate Division

by

Peter Warning Cowling

In Partial Fulfillment

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SUMMARY

The motivation for this study was an effort to find some means of improving the performance of a thermoelectric heat pump other than through the development of new semiconductor materials. One of the means employed was the optimization of a standard cascaded thermoelectric heat pump. For a given hot and cold junction temperature the coefficient of performance (C.O.P.) of a cascaded thermoelectric heat pump is a function of the interstage temperatures. Several of the most rigorous works to date which have tried to find the optimal interstage temperatures were based on a simplifying assumption to the governing equation which had the far-reaching effect of forcing the C.O.P. of each stage to be equal for the maximum C.O.P. of the cascaded heat pump. However, without making this simplifying assumption, the governing equation was so unwieldy that it could not be optimized in closed form. Therefore, optimization was carried out numerically and the results showed that the stage temperatures at optimal C.O.P. were given by the following equation for an n-stage cascade:

$$\frac{T_2}{T_1} = \frac{T_3}{T_2} = \ldots\ldots = \frac{T_n}{T_{n-1}} .$$

With the above interstage temperatures the stage C.O.P.'s are not equal, but the optimal C.O.P. for the entire cascade is nearly the same as that determined by the simplified method. Thus, it is concluded that the overall C.O.P. is relatively insensitive to small changes in the interstage

temperatures.

In many cases a cascade system offers only a very slight gain in C.O.P. over a single-stage system operating under the same conditions, so that the added cost and difficulty of fabrication of a cascade system is probably not justified. As a design guide, a simplified, approximate equation was developed which gives the ratio of the C.O.P. of an n-stage cascaded thermoelectric heat pump to that of a single-stage thermoelectric heat pump.

An energy balance at the cold junction shows that, for a typical case of a single-stage thermoelectric heat pump, more than 75 per cent of the Peltier heat removed from the cold junction originated from Joule heating and heat conduction from the hot junction. This motivated the development of a modified cascade thermoelectric heat pump which reduced the temperature gradient in the legs of the element and also reduced the Joule heating in the area of the cold junction. By this means some of the current which passed through the hot junction was shunted by the cold junction thereby reducing the cold junction current flow. This shunted current also removed heat due to Peltier cooling and thus reduced the temperature gradient in the legs of the element at the cold junction.

The governing equation for the modified cascade thermoelement was developed and found to be a function of the following parameters:

(a) The physical properties of the semiconductor elements.

(b) The ratio of length to cross-sectional area of the elements.

(c) The hot and cold junction temperatures.

(d) The hot and cold junction currents.

(e) The location of the electrical tap.

In order to optimize the governing equation, it was necessary that values be assumed for the physical properties of the semiconductor, and the hot and cold junction temperatures. The equation was investigated by numerical analysis to find the optimal value. The optimization showed that the C.O.P. was not affected by the ratio of length to cross-sectional area of the elements, but the cooling capacity increased proportionately with this ratio. These results are the same as those found for a single-stage thermoelectric heat pump.

Since the governing equation for the modified cascade thermoelectric heat pump could not be solved in closed form, some results for a typical case are shown below, comparing it with a single-stage thermoelectric heat pump and a two-stage cascade thermoelectric heat pump. The properties assumed for the semiconductor are typical of those found in materials currently available. All three cases were solved by numerical methods for a hot junction temperature of 300°K and a cold junction temperature of 250°K.

The results for a modified cascade heat pump with electrical tap located six-tenths of the way from the hot junction to the cold junction, are as follows:

C.O.P. = 0.2205 (maximum value)

Q_c = 0.356 watts (at maximum C.O.P.)

C.O.P. = 0.110 (at maximum Q_c)

Q_c = 0.639 watts (maximum value)

The results for a single-stage thermocouple with semiconductor elements of the same size as those in the modified cascade are:

C.O.P. = 0.158 (maximum value)

Q_c = 0.227 watts (at maximum C.O.P.)

C.O.P. = 0.1225 (at maximum Q_c)

Q_c = 0.300 watts (maximum value)

Finally the results for a two-stage cascade based on the same cooling capacity as the modified cascade thermocouple operating at maximum C.O.P. are:

Stage 1 C.O.P. = 0.717

Stage 2 C.O.P. = 0.920

Overall C.O.P = 0.251

Q_c = 0.356 watts (Stage 1)

Q = 0.836 watts (Stage 2)

Thus it can be seen for this example that the modified cascade thermocouple operates with a C.O.P. about 40 per cent higher than that of a single-stage thermocouple and about 13 per cent lower than the maximum C.O.P. of a two-stage cascade. It has an advantage over the two-stage cascade inasmuch as it has a far greater range of cooling capacity. If the two-stage cascade is designed to operate at maximum C.O.P., the cooling capacity can only be increased very slightly above that obtainable at maximum C.O.P. This is due to the fact that the second stage must remove all of the work supplied to the first stage as well as Q_c, and as the C.O.P. of the first stage decreases, the load on the second stage increases by $\Delta Q_c + \frac{\Delta W_1}{\text{C.O.P.}_1 + \Delta(\text{C.O.P.}_1)}$.
Thus the load limit of the second stage is reached very quickly, thereby limiting the cooling capacity of the cascade thermocouple.

In order to verify the theory of the modified cascade heat pump an experimental model was made. In fabrication of the model contact resistances of the order of 0.0003 ohms were formed at the soldered junctions. Although these are small, it is more than 10 per cent of the element resistance and thus a significant factor. Also a heat leak was found to exist from the surroundings to the cold junction. When the analytical treatment was extended to account for these factors, a reasonable correlation with the analytical curves was obtained. The analytical treatment was also somewhat in error since the physical properties were assumed to be constant at those values measured at 300°K. Considering these discrepancies the correlation appears to be sufficiently close to substantiate the theory of the modified cascade thermoelectric heat pump.

NOMENCLATURE

Symbol		Typical Units
A	cross-sectional area of semiconductor element	cm^2
C.O.P.	coefficient of performance of a heat pump	dimensionless
c_1,c_2	hot junction contact resistance	ohms
c_2	center tap contact resistance	ohms
c_3	cold junction contact resistance	ohms
I	current	amps
I_1	hot junction current of modified cascade thermocouple	amps
I_2	center tap current of modified cascade thermocouple	amps
I_3	cold junction current of modified cascade thermocouple	amps
K	thermal conductance	watts/°K
K_1	thermal conductance of element on hot junction side of center tap	watts/°K
K_2	thermal conductance of element on cold junction side of center tap	watts/°K
°K	degrees Kelvin	
L	length of semiconductor element	cm
n	electron 'donor' type element (electron or negative type)	
p	electron 'acceptor' type element (hole or positive type)	
Q	heat	watts

Symbol		Typical Units
Q_c	heat removed at cold junction of thermocouple	watts
R	electrical resistance	ohms
R_1	electrical resistance of element on hot junction side of center tap	ohms
R_2	electrical resistance of element on cold junction side of center tap	ohms
T	temperature	$^\circ K$
T_c	cold junction temperature	$^\circ K$
T_h	hot junction temperature	$^\circ K$
T_m	mean temperature of hot and cold junctions	$^\circ K$
T_3	center tap temperature of modified cascade thermocouple	$^\circ K$
V	electrical potential	volts
W	input energy to heat pump	watts
x	distance from hot junction of thermocouple	cm
x_1	distance of center tap from hot junction of thermocouple	cm
Z	figure of merit for a thermocouple	$1/^\circ K$

Greek Symbols

α_{np}	Seebeck Coefficient at a junction of 'n' and 'p' type elements	volts/$^\circ K$
γ_n	Thomson Coefficient for a 'n' type element	watts/amps$^\circ K$
ΔT	temperature difference between hot and cold junctions of heat pump	$^\circ K$
ζ	ratio of net heat removal to gross heat removal at cold junction	dimensionless
κ	thermal conductivity	watts/$^\circ K$ cm

Symbol		Typical Units
π	repeated product	
π_{np}	Peltier Coefficient at a junction of 'n' and 'p' type elements	watts/amp
ρ	electrical resistivity	ohm cm

Subscripts

i,n	integers denoting stage of multi-stage thermocouples
max	maximum
opt	optimal

CHAPTER I

INTRODUCTION

Background

The phenomenon of thermoelectric cooling was first noted by Peltier in 1834. He observed that passing a current through a junction of two dissimilar conductors resulted in the absorption or rejection of heat, depending upon the direction of current flow. In honor of his discovery a Peltier coefficient has since been defined by*

$$\pi_{pn} = \frac{Q}{I} \tag{1}$$

where p and n represent the two dissimilar conductors. Seebeck had found earlier, in 1821, that it was possible to produce a current by heating the junctions between dissimilar conductors. This relationship is known as the Seebeck effect, the Seebeck coefficient being defined by

$$\alpha_{pn} = \lim_{\Delta T \to 0} \frac{\Delta V}{\Delta T} \tag{2}$$

where ΔV is the potential difference between the two junctions of conductors p and n and ΔT is the temperature difference between these junctions. A third effect is the Thomson effect, which states that if a current passes through a conductor in the direction of a temperature gradient, then a quantity of heat, ΔQ, is generated.

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*Symbols are defined on page xiii.

Next the Thomson coefficient is defined

$$\gamma = \lim_{\Delta T \to 0} \frac{\Delta Q}{I \Delta T} . \tag{3}$$

Notice that both the Peltier and Seebeck effects are the results of junctions between two dissimilar conductors. However, the Thomson effect occurs in a single conductor with current flowing in a temperature gradient.

Kelvin applied the laws of thermodynamics to a simple circuit and found the following relationships, which are named after him (1);*

$$\alpha_{np} = \frac{\pi_{np}}{T} , \tag{4}$$

$$\frac{d\alpha_{np}}{dT} = \frac{\gamma_n - \gamma_p}{T} . \tag{5}$$

Thus it can be seen that the Seebeck, Peltier, and Thomson coefficients are temperature dependent. If $\gamma_n = \gamma_p$, then the Seebeck coefficient is independent of temperature and the following relationship holds for the Peltier coefficient:

$$\pi_{np} = \text{constant} \times T .$$

Consider a simple thermocouple such as the one shown in Figure 1.

The Thomson cooling for the entire element is given by

$$I \oint_{loop} \gamma \, dT = I \int_{T_c}^{T_h} (\gamma_n - \gamma_p) \, dT .$$

*Numbers within parentheses refer to items listed in the Bibliography.

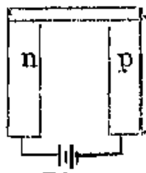

Figure 1. Simple Thermocouple.

But from Kelvin's second relationship,

$$(\gamma_n - \gamma_p)dT = T\, d\,\alpha_{np} .$$

Thus the Thomson cooling may be written as $I\int_{T_c}^{T_h} T\, d\,\alpha_{np}$. Since the conduction heat transfer equation is a linear differential equation, one-half of the Thomson cooling appears at each junction. So at the cold junction the total cooling is given by the sum of the Peltier cooling and one-half the Thomson cooling and can be expressed by

$$Q = IT_c\alpha_{np_{T_c}} + \frac{1}{2} I \int_{T_c}^{T_h} T\, d\,\alpha_{np} . \tag{6}$$

It is known that the Peltier term is much larger than the Thomson term, and the integral can therefore be approximated without seriously altering the value of Q by the following method:

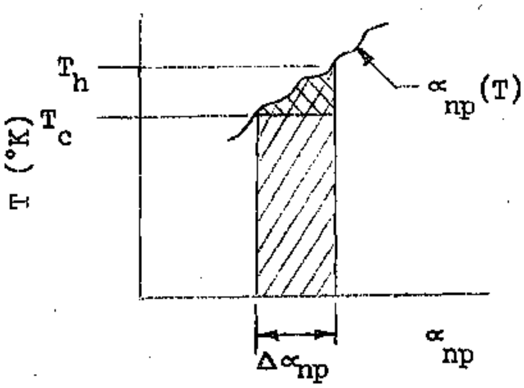

Figure 2. Seebeck Coefficient as a Function of Temperature.

As an approximation to the integral, consider the shaded area of Figure 2 which can be approximated by

$$\int_{T_c}^{T_h} T d\alpha_{np} \cong T_c(\Delta\, \alpha_{np}) = T_c \left[\alpha_{np_{T_h}} - \alpha_{np_{T_c}}\right] . \tag{7}$$

Now the heat removal is given by

$$Q = IT_c\, \alpha_{np_{T_c}} + \frac{1}{2}\, IT_c \left[\alpha_{np_{T_h}} - \alpha_{np_{T_c}}\right],$$

or

$$Q = IT_c \left[\frac{\alpha_{np_{T_h}} + \alpha_{np_{T_c}}}{2}\right] . \tag{8}$$

But the bracketed quantity is just the average value for α_{np} based on the junction temperatures. Thus if α_{np} is evaluated at the average temperature, the heat removal can be written,

$$Q_c = \alpha_{np_{T_m}}\; IT_c , \tag{9}$$

where T_m is defined as $\frac{T_h + T_c}{2}$.

By this means it is possible to account for the Thomson cooling without evaluating the integral shown in Equation (6).

At the same time that heat is being removed from the cold junction by the combined Peltier and Thomson effects, heat is also being deposited at this junction due to Joule heating and thermal conduction from the hot junction. Again, since the conduction heat transfer equation is a linear differential equation, one half the Joule heat finds its way to each junction. Thus, if Q_c is defined as the net heat removed from the cold junction, then it can be expressed mathematically as,

$$Q_c = \alpha_{np} T_c I - \frac{1}{2} I^2 R - K(T_h - T_c) , \tag{10}$$

where

$$R = \frac{\rho_n L_n}{A_n} + \frac{\rho_p L_p}{A_p} \tag{11}$$

and

$$K = \frac{\kappa_n A_n}{L_n} + \frac{\kappa_p A_p}{L_p} . \tag{12}$$

The chief criterion for the evaluation of a heat removal device is the ratio of heat removal to energy expended in removing the heat. This ratio is the coefficient of performance and described mathematically as

$$C.O.P. = \frac{Q_c}{W} . \tag{13}$$

The energy supplied to the couple consists of two parts:

1. The Joule heating.
2. The energy to overcome the Seebeck voltage resulting from the different temperatures of the junctions. This energy is expressed as $\alpha_{np} I \Delta T$.

Thus the energy expended in removing Q_c is expressed as,

$$W = I^2 R + \alpha_{np} I \Delta T . \tag{14}$$

And now the coefficient of performance can be written,

$$C.O.P. = \frac{\alpha_{np} T_c I - \frac{1}{2} I^2 R - K \Delta T}{I^2 R + \alpha_{np} I \Delta T} . \tag{15}$$

For any given semiconductor material and source and sink temperatures, the C.O.P. is a function of the couple current, I, and the geometry of the legs, L_n/A_n and L_p/A_p. If the thermocouple is optimized under these conditions, it is found that*

$$C.O.P._{max} = \frac{T_m}{\Delta T} \frac{\sqrt{1 + ZT_m} - 1}{\sqrt{1 + ZT_m} + 1} - \frac{1}{2} , \tag{16}$$

where

$$T_m = \frac{1}{2} (T_c + T_h) ,$$

and Z is

$$Z = \frac{\alpha_{np}^2}{[(\rho_p K_p)^{1/2} + (\rho_n K_n)^{1/2}]^2} . \tag{16a}$$

By defining Z it is possible to reduce the C.O.P. to a function of only one parameter. Notice that as the value of Z increases, the C.O.P. increases. Hence Z is called the figure of merit of the thermocouple.

In order to investigate the heat removal by a thermocouple consider a parameter ζ with the definition

$$\zeta = \frac{Q_c}{Q_{Peltier}} = \frac{\alpha_{np} T_c I - \frac{1}{2} I^2 R - K \Delta T}{\alpha_{np} T_c I} . \tag{17}$$

*See Appendix I.

Thus ζ is a ratio of the net heat removal to the gross heat removal at the cold junction of the thermocouple. When the couple is operating at maximum C.O.P. between any two temperatures then

$$\zeta_{opt} = 1 - \frac{1}{2}\frac{\Delta T}{T_c}\frac{1}{\sqrt{1 + ZT_m} - 1} - \frac{\sqrt{1 + ZT_m} - 1}{ZT_c}, \qquad (18)$$

A graph of ζ_{opt} vs. Z is shown in Figure 3 for a source temperature of 250°K and a sink temperature of 300°K. Since most known thermoelectric materials have a figure of merit between 0.002 and 0.003, it is interesting to note that more than 75 per cent of the heat removed is that due to Joule heating and heat conduction from the hot junction. Thus it would appear that if steps could be taken to reduce either or both of these effects, the performance of the couple would be improved. It was this fact that motivated the development of the modified cascade system in Chapter III.

Review of Literature

Even though the discovery of Peltier cooling is over two hundred years old, thermoelectric cooling was impractical until the development of semi-conductor materials. Today thermoelectric refrigeration is still not practical for many applications. This is due to the fact that, with any of the known semiconductors, it is difficult to get a C.O.P. exceeding 2 with a temperature difference exceeding 20°K. Until recently most research concerned with improving the C.O.P. dealt solely with the development of new semi-conductors. Presently this field of development has reached a plateau and thus other means of improvement must be sought.

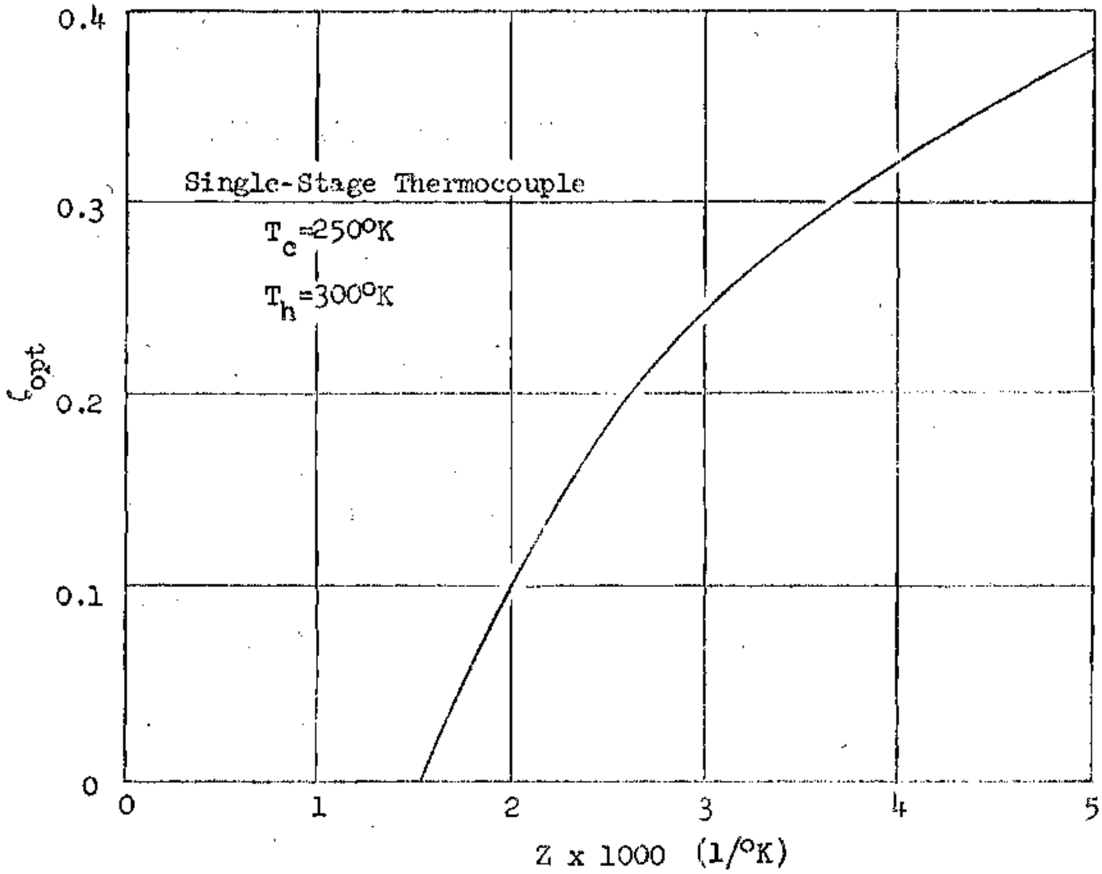

Figure 3. ζ_{opt} as a Function of Z for a Single-Stage Thermocouple.

Various methods have been employed including variable cross-sectional area of the elements (2) and constructing elements utilizing surface heat transfer (3). Both methods increase the difficulty of fabrication of the thermal elements into a useful device and consequently have not undergone much development.

A method which may have more promise involves cascading the elements. Presently some work has been done in this area, but much of it tends to be more speculative than analytical.

Foster (4) considers a two-stage cascade with the following variations:

(a) The cascade has coupled stages of n and $n+1$ couples respectively. (Stages are parallel electrically and series thermally.)

(b) The cascade has two stages which are in series electrically and thermally.

From a computer solution he found that for optimum C.O.P.

$$\frac{T_m}{T_h} = \frac{T_c}{T_m}$$

where T_m = interstage temperature. The coupled stage cascade was found to be superior to the electrically insulated cascade. However, the insulated cascade was more easily fabricated.

Jaumet (5), in an article written in 1958, concludes that cascading offers its greatest promise in refrigeration. He further states that cascading lowers the load capacity of each stage so drastically that it has not yet proved practical to use more than two stages.

Crump (6) states that by cascading, temperature differences of 100°C can be obtained. In cascading he postulates that, as a rule of thumb, each stage should have from three to three and one-half times as many junctions as the preceding stage. These stages are wired electrically in series and have equal and constant cross sectional areas.

O'Brien, Wallace, and Landecker (7) and Goldsmid (1) both deal with two-stage cascades in which both stages operate with equal temperature differences and C.O.P. Their results showed improvement over single stage performance.

Ioffe (8) states that for the best utilization of all stages of a cascade system, each stage of the cascade should be "of sharply decreasing power." (No figures are given as to how sharp this decrease should be.) He develops an expression for a cascade system which gives the overall C.O.P. as a function of the C.O.P.'s of each stage as follows:

$$1 + \frac{1}{C.O.P.} = \prod_{i=1}^{i=n} \left(1 + \frac{1}{C.O.P._i}\right).$$

The most sophisticated treatment mathematically seems to be by Rittner (9). He makes an approximation for one of the terms in the repeated product and shows that under optimal conditions each stage has the same C.O.P. (This is discussed further in Chapter III.)

Clingman (10) used irreversible thermodynamics and an entropy analysis to develop temperature dependent integral expressions for the C.O.P. and figure of merit for a thermoelectric device. From this he is able to develop an expression for a cascade of infinite stages. However, in order to obtain a solution to a particular problem he is forced to assume that the figure of merit is constant and to insert specific values for the properties of the elements. This seems to be a common dilemma encountered by most researchers as it is not always convenient to non-dimensionalize the equations, and it is necessary to deal with a specific example rather than the general case.

Other articles (11 - 27) found on thermoelectric cooling seemed to have no direct bearing on this study.

CHAPTER II

CASCADES

Maximum Coefficient of Performance

The net heat removal by a thermocouple is given by equation (10)

$$Q_c = \alpha_{np} T_c I - \frac{1}{2} I^2 R - K \Delta T \ . \tag{10}$$

Obviously if ΔT becomes large enough the heat removal can be made to approach zero. Consequently the coefficient of performance also approaches zero, and from equation (16) is obtained

$$\Delta T_{max} = 2T_m \frac{\sqrt{1 + ZT_m} - 1}{\sqrt{1 + ZT_m} + 1} \ . \tag{19}$$

Thus, with any single stage thermocouple, there exists a maximum temperature difference across which heat can be removed. Therefore, as ΔT approaches ΔT_{max}, the C.O.P. approaches zero. In order to circumvent this situation thermocouples are placed thermally in series in such a way that the hot junction of the first couple becomes the cold junction of the second couple, and so on. This practice is known as cascading.

The coefficient of performance for a couple of n stages is given by Ioffe (8)

$$1 + \frac{1}{C.O.P.} = \prod_{i=1}^{n} \left[1 + \frac{1}{C.O.P._i} \right] , \tag{20}$$

where π represents a repeated product. Now if each stage is designed so that it is operating at its peak efficiency, equation (16) can be substituted into equation (20) with the following result:

$$1 + \frac{1}{C.O.P.} = \prod_{i=1}^{n} \frac{2T_{m_i}(\sqrt{1 + ZT_{m_i}} - 1) + \Delta T_i(\sqrt{1 + ZT_{m_i}} + 1)}{2T_{m_i}(\sqrt{1 + ZT_{m_i}} - 1) - \Delta T_i(\sqrt{1 + ZT_{m_i}} + 1)} \quad . \quad (21)$$

In terms of the interstage temperatures,

$$1 + \frac{1}{C.O.P} = \prod_{i=1}^{n} \frac{T_i \sqrt{1 + \frac{Z}{2}(T_i + T_{i-1})} - T_{i-1}}{T_{i-1}\sqrt{1 + \frac{Z}{2}(T_i + T_{i-1})} - T_i} \quad . \quad (22)$$

When C.O.P. is maximum, $1 + \frac{1}{C.O.P.}$ is at a minimum. Thus to find the interstage temperatures which give the optimal performance of the cascade,

$$\frac{\partial(1 + \frac{1}{C.O.P.})}{\partial T_i} = 0 \quad \text{for} \quad 1 \leq i \leq n + 1 \quad (23)$$

where T_i = cold junction temperature

T_{n+1} = hot junction temperature.

From equation (23) n-1 simultaneous equations can be obtained which, upon solution, yield the interstage temperatures. The actual solution of these equations would be extremely difficult and for this reason the extremal values were found by a numerical optimization process on a digital computer.* The optimization was done for a system which had a source temperature of 250°K and a sink temperature of 300°K, and for systems with 2, 3, and 4 stages. The interstage temperatures are tabulated in Tables 1, 2, and 3.

From these results it is possible to conclude that the interstage temperatures are practically insensitive to the figure of merit. In other words, at optimal conditions, the same interstage temperature will be found for any realistic value of the figure of merit. Also notice what happens when the following equation is solved for each of the systems.

$$\frac{T_2}{T_1} = \frac{T_3}{T_2} = \ldots\ldots \frac{T_{n+1}}{T_n} \qquad (24)$$

For two stages this yields,

$$\frac{T_2}{250} = \frac{300}{T_2}$$

and

$$T_2 = 273.86°K \; .$$

This compares very favorably with the values found for T_2 in Table 1. Similarly for three stages equation (24) yields

$$\frac{T_2}{250} = \frac{T_3}{T_2} = \frac{300}{T_3} \; .$$

*See Appendix 2.

Table 1. C.O.P. for Two Stages

$Z \times 10^3$	T_2 (°K)	C.O.P.
1	272.8	0.02014
2	273.0	0.20167
3	273.0	0.39332
4	273.0	0.56596
5	273.0	0.71927

Table 2. C.O.P. for Three Stages

$Z \times 10^3$	T_2 (°K)	T_3 (°K)	C.O.P.
1	265.0	281.8	0.03751
2	264.8	281.4	0.21743
3	265.0	281.6	0.40555
4	265.0	281.6	0.57597
5	265.0	281.6	0.72780

Table 3. C.O.P. for Four Stages

$Z \times 10^3$	T_2 (°K)	T_3 (°K)	T_4 (°K)	C.O.P.
1	261.2	273.4	286.0	0.04357
2	261.0	273.0	286.0	0.22280
3	261.0	273.0	286.0	0.40977
4	261.0	273.0	286.0	0.57944
5	261.2	273.6	286.6	0.73075

Upon solution this equation yields

$$T_2 = 265.6°K$$

and

$$T_3 = 282.3°K\ .$$

Again this compares favorably with the values found in Table 2. Next for four stages equation (24) yields

$$\frac{T_2}{250} = \frac{T_3}{T_2} = \frac{T_4}{T_3} = \frac{300}{T_4}\ .$$

Solution of the above relationships yields

$$T_2 = 261.6°K$$

$$T_3 = 273.8°K$$

and

$$T_4 = 286.6°K\ .$$

Once more these values agree well with those found in Table 3.

Therefore, on the basis of the information given in Tables 1, 2, and 3, the optimal value for the C.O.P. of a cascade of n stages is obtained when the following relationship exists between the interstage temperatures;

$$\frac{T_2}{T_1} = \frac{T_3}{T_2} = \ldots\ldots = \frac{T_{n+1}}{T_h}\ .$$

As it turns out the C.O.P. does not vary much as the interstage temperature is varied about its optimal value. This fact is illustrated for a two stage cascade in Figure 4.

Rittner (9) optimized equation (21) for the condition that $\sqrt{1 + ZT_{m_i}}$ could be considered constant. The eventual result of this assumption is the C.O.P. of every stage is the same under optimal interstage temperatures. This reduces the repeated product of equation (20) to the following power form for an n-stage cascade:

$$1 + \frac{1}{\text{C.O.P.}} = \left(1 + \frac{1}{\text{C.O.P.}_n}\right)^n ,$$

where C.O.P._n is the stage C.O.P. At first glance this would seem to be erroneous. For example, if

$$T_c = 250°K$$

$$T_h = 300°K$$

and

$$Z = 2 \times 10^{-3} \frac{1}{°K}$$

for a two-stage thermocouple, the following occurs at optimal conditions. For the actual case the stage C.O.P.'s are

$$\text{C.O.P.}_1 = 0.651$$

and

$$\text{C.O.P.}_2 = 0.738 .$$

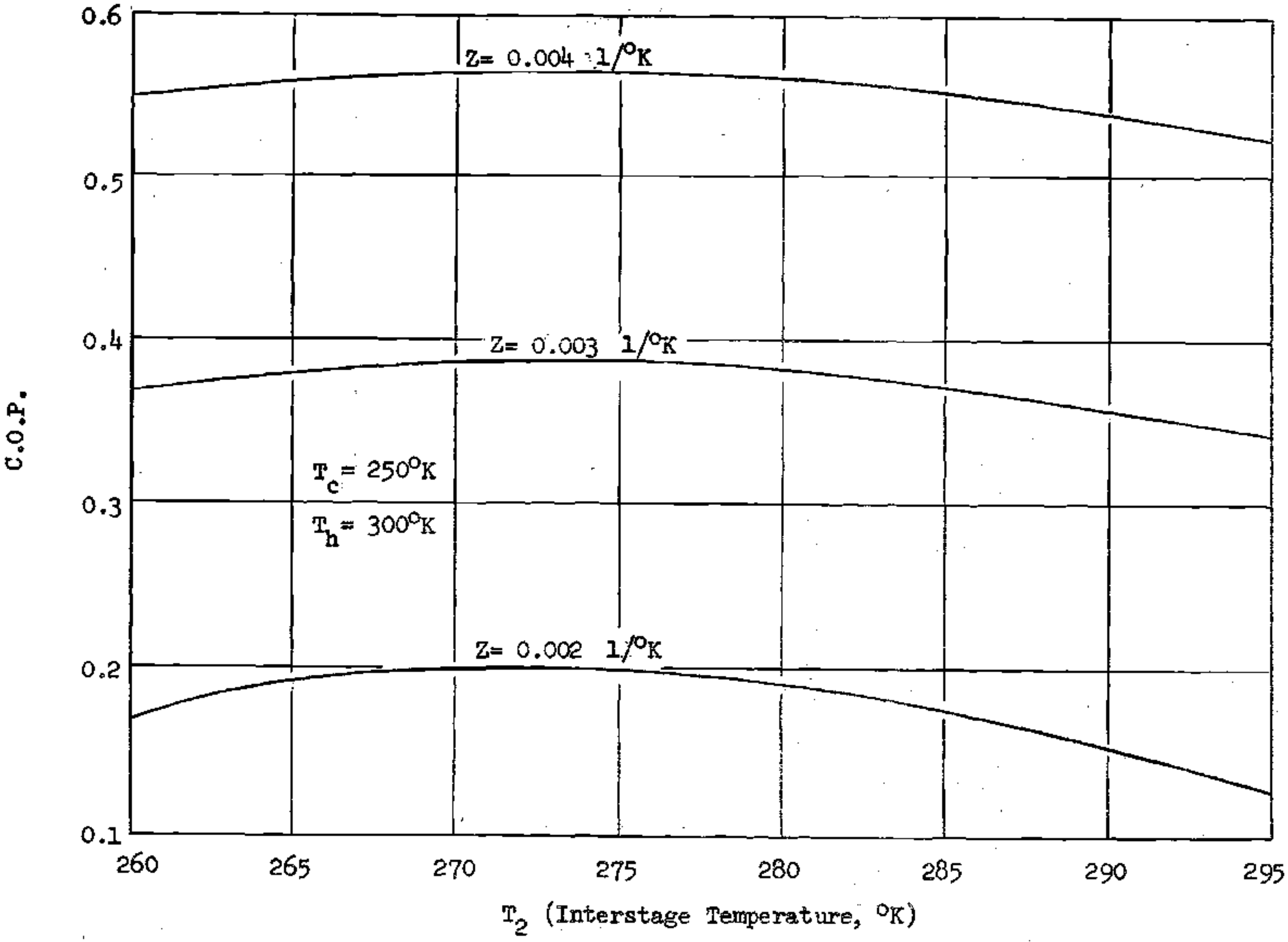

Figure 4. C.O.P. vs. Temperature for a Two-Stage Cascade.

(A variation of greater than thirteen per cent in the stage C.O.P.'s.) For Rittner's case, where the stage C.O.P.'s are the same, the C.O.P.'s are 0.695. However, when the overall C.O.P. for the cascade is found from equation (20) the actual value and Rittner's value are essentially the same.

One of the large disadvantages of cascading is that each successive stage must remove all of the heat resulting from the work done on the preceding stages, as well as the heat removed from the source. For example, suppose the cascade has a C.O.P. of φ per stage and removes an amount of heat, Q_c, from the cold junction. The input energy to the first stage is

$$W_1 = \frac{Q_c}{\varphi} .$$

Now the second stage must remove $Q_c + \frac{Q_c}{\varphi}$, $Q_1 + W_1$, from the first stage. For the third stage the heat removal is

$$Q_3 = Q_2 + W_2 = Q_c[1 + \frac{1}{\varphi}] + \frac{Q_2}{\varphi} = Q_c[1 + \frac{1}{\varphi}]^2 .$$

and for n-stages

$$Q_n = Q_c[1 + \frac{1}{\varphi}]^n .$$

So, even though cascading does improve the C.O.P., the increased cost of materials and the difficulty of fabrication soon tend to overshadow this improvement. Also there is a rapidly diminishing gain in the C.O.P. as successive stages are added to the system. This is illustrated for a typical case in Figure 5. Coupled with this, as the figure of merit (Z) increases, there is not much to be gained by any type of cascading as when compared to a single stage device. More simply stated this says

$$\lim_{Z \to \infty} \frac{\text{C.O.P.}_{\text{cascade}}}{\text{C.O.P.}_{\text{single stage}}} \to 1.$$

This is pointed out for a typical case in Figure 6.

For these reasons, it would be advantageous to develop an expression which would easily yield a value for $\frac{\text{C.O.P.}_{\text{single stage}}}{\text{C.O.P.}_{\text{cascade}}}$, without going through rigors of solving the repeated product expression in equation (20). Since the C.O.P. decreases as $\Delta T \to \Delta T_{max}$, it would seem that ΔT_{max} would be a good parameter to consider in developing this expression. An approximate expression with three per cent accuracy is given by *

$$\frac{\text{C.O.P.}_{\text{single stage}}}{\text{C.O.P.}_{\text{cascade}}} \cong \frac{2\Delta T}{\Delta T_{max} - \Delta T} \frac{(\Delta T_{max} - \frac{1}{n}\Delta T)^n}{(\Delta T_{max} + \frac{1}{n}\Delta T)^n - (\Delta T_{max} - \frac{1}{n}\Delta T)^n} \quad (25)$$

where ΔT_{max} is given by equation (19).

Maximum Heat Removal by a Two-Stage Couple

Suppose, for any given two stage thermocouple, the rate of heat removal is increased for both stages. As the rate of heat removal increases, the cooling capacity of the second stage limits the overall heat

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*See Appendix 3.

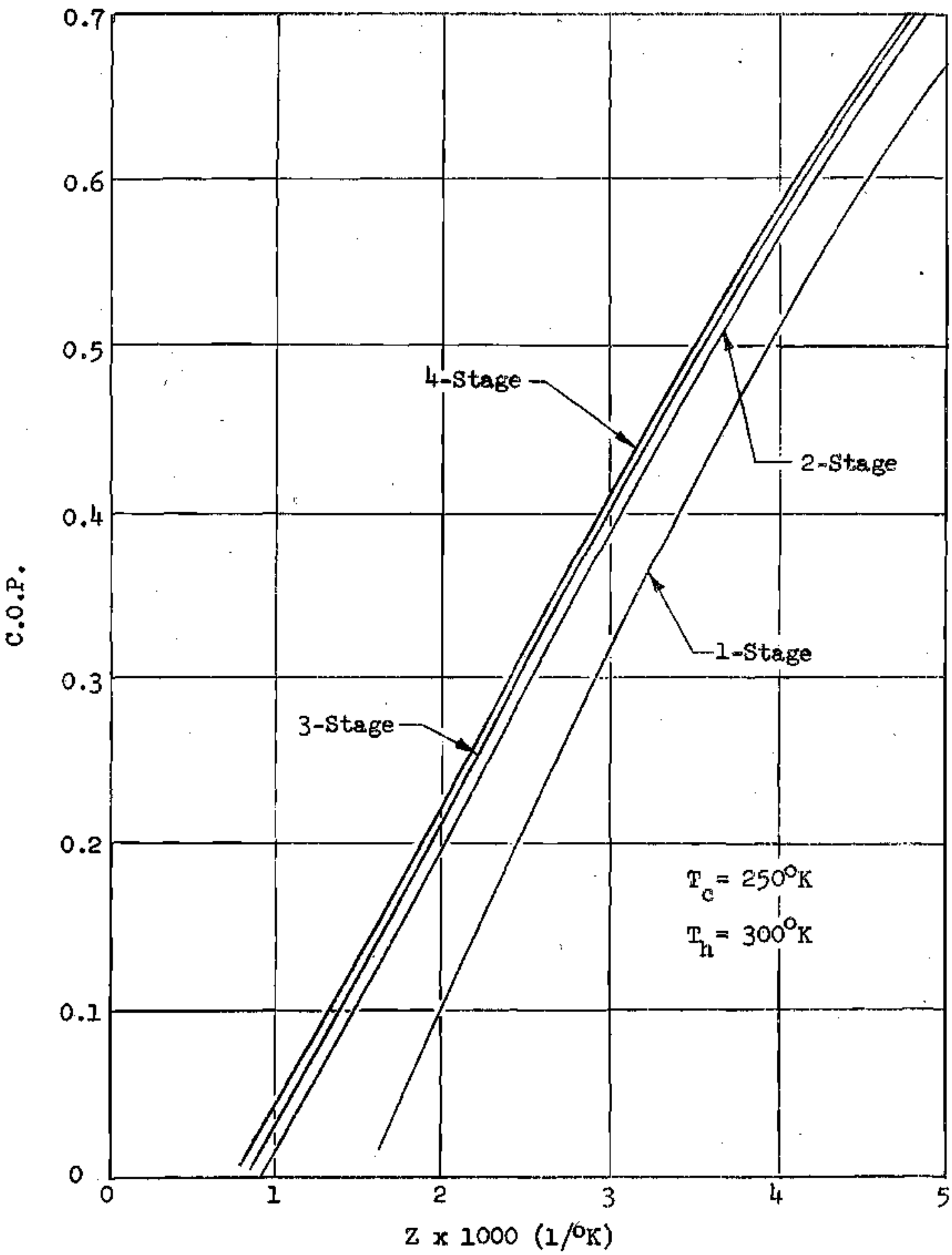

Figure 5. Z vs. C.O.P. for Various Staging of Cascades.

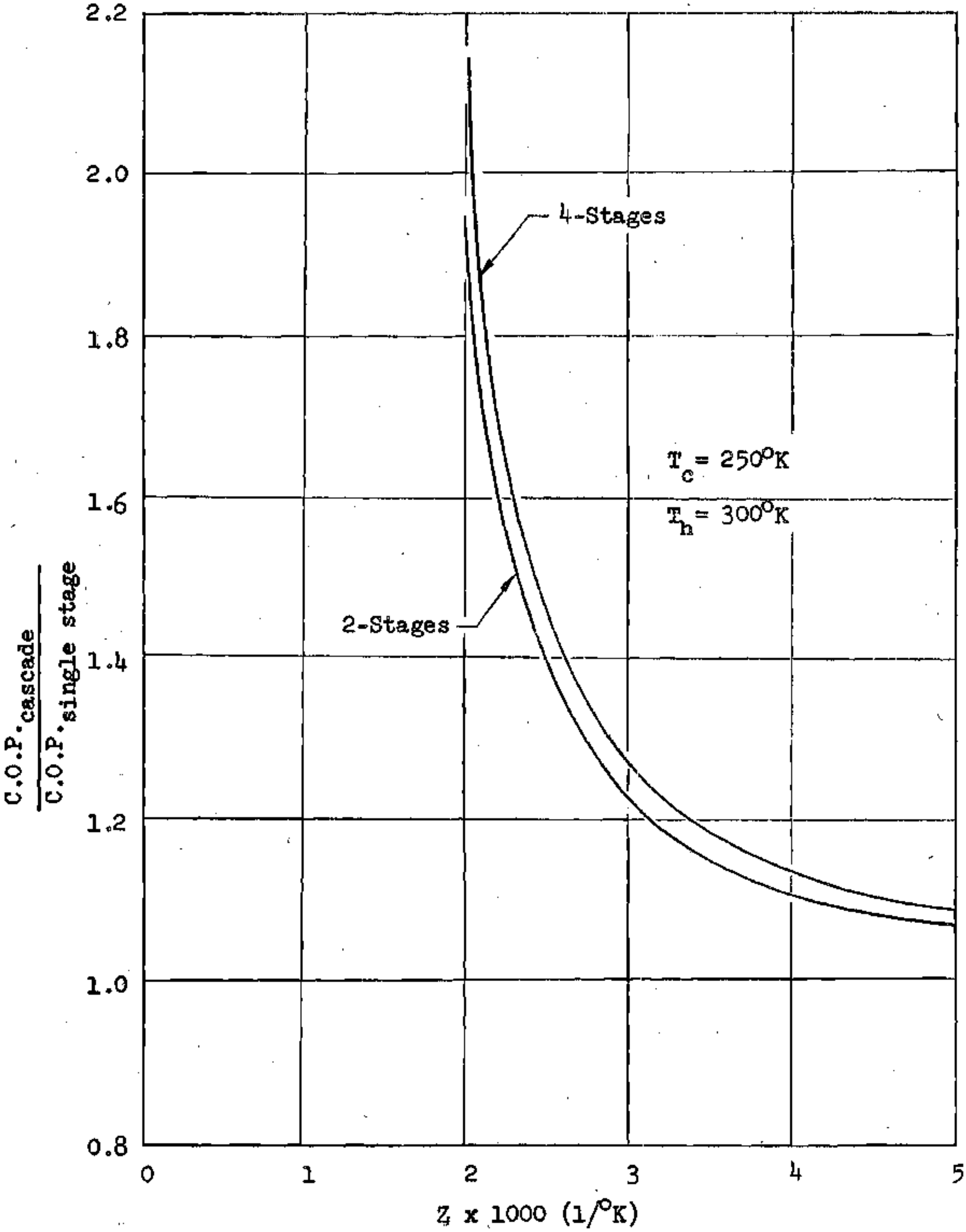

Figure 6. $\frac{\text{C.O.P.}_{\text{cascade}}}{\text{C.O.P.}_{\text{single stage}}}$ **vs. Z for 2- and 4-Stage Cascades.**

removal of the thermocouple.* This limitation suggests that the second stage be built with more cooling capacity than is required under normal operating conditions. Thus under less than peak cooling loads the second stage will cool the interstage junction until it has the same temperature as the cold junction, and the first stage operates with $\Delta T = 0$. At this point the C.O.P. of the first stage approaches infinity. Now from equation (20) the overall C.O.P. can be written,

$$1 + \frac{1}{\text{C.O.P.}} = \left(1 + \frac{1}{\text{C.O.P.}_2}\right)\left(1 + \frac{1}{\infty}\right).$$

The above equation reduces to

$$\text{C.O.P.} = \text{C.O.P.}_2 .$$

This reduces the device to a single-stage device under normal operating conditions.

So that while a two-stage device gives an improvement in C.O.P. while operating at the optimal cooling rate, any increase in this rate cannot be tolerated due to the escalated effect on the second stage.

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*See Appendix 4.

CHAPTER III

THE MODIFIED CASCADE

Development and Optimization of Coefficient of Performance

It was pointed out earlier that a typical single-stage thermocouple uses seventy-five per cent of its Peltier cooling just to remove Joule heating and conduction from the hot junction. This tends to make a low C.O.P. inherent with the system. A standard system of cascades will improve the C.O.P., but it was shown that, in a typical case, each stage must have roughly three times the cooling capacity of the preceding stage.

In an effort to improve on both the cascade and single-stage systems consider the system shown in Figure 7. By the insertion of the center tap, part of the current, I_2, does not pass through the cold junction and thereby reduces the Joule heat that must be removed from that junction. Also Peltier heat is removed from the legs at the center tap, and thus less heat is conducted from the hot junction back to the

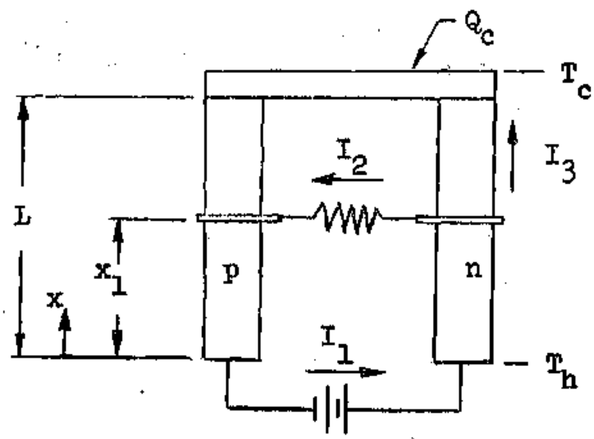

Figure 7. Modified Cascade Thermocouple.

cold junction. Both of these effects are achieved at the loss of Peltier cooling at the cold junction since $I_3 \alpha_{np} T_c$ is less than $I_1 \alpha_{np} T_c$. But it is hoped that some optimal conditions exist at which an improved level of performance over a simple thermocouple can be obtained.

The modified cascade is primarily different from conventional cascades in the respect that it has one leg spanning the entire gap from the cold junction to the hot junction. (The center tap serves to reduce the temperature gradient in this leg and to bypass some of the current from passing through the cold junction.) An analogous representation of the modified cascade thermocouple is shown in Figure 8.

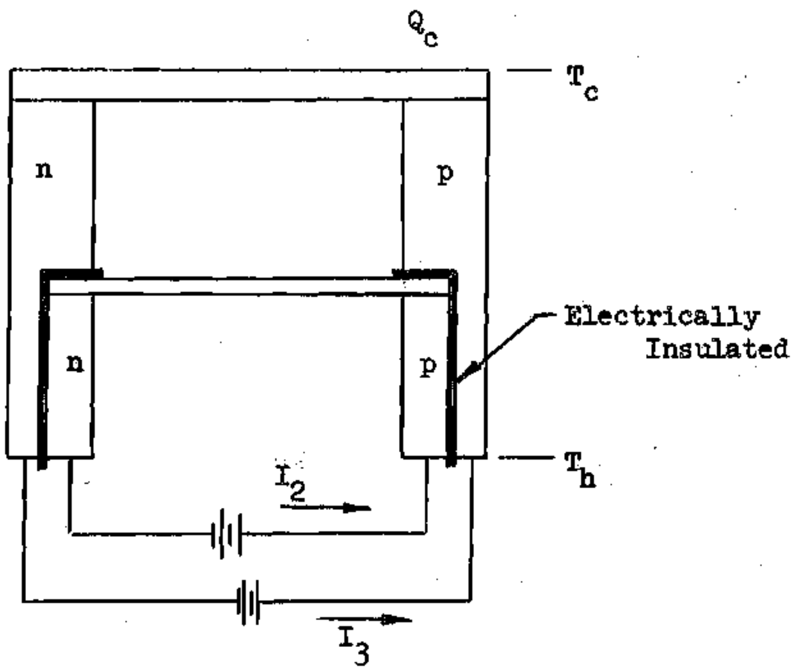

Figure 8. Representation of a Modified Cascade Thermocouple.

The governing equations for this modified cascade system are developed in Appendix 5. It is assumed that the thermocouple properties are known as well as the sink and source temperatures. Thus the C.O.P.

becomes a function of I_1, I_2, I_3 and the geometry of the element. From circuit considerations and an energy balance at the center tap it can be reduced to the following expression*

$$
\begin{aligned}
\text{C.O.P.} = \Bigg\{ & \alpha_{np} T_c I_3 - 2\,\frac{A}{L}\Big[I_3^2 \frac{\rho L}{A}\left(1 - \frac{x_1}{L}\right) + I_1^2\,\frac{\rho L}{A}\left(1 - \frac{x_1}{L}\right) \\
& + 2K\frac{A}{L}\,\frac{T_c}{1 - \frac{x_1}{L}} + 2K\frac{A}{L}\,\frac{T_h}{1 - \frac{x_1}{L}}\Big] \div \left[1 - \frac{x_1}{L}\right]\Big[\alpha_{np}(I_1 - I_3) \\
& + \frac{2KAL}{x_1(L - x_1)}\Big]\Bigg\} \div \Bigg\{ 2\,\frac{\rho L}{A}\left(\frac{x_1}{L}\right) I_1^2 + 2\,\frac{\rho L}{A}\left(1 - \frac{x_1}{L}\right) I_3^2 \\
& + 2K\,\frac{A}{L}\,\frac{T_c}{L\left(1 - \frac{x_1}{L}\right)} - \frac{\rho L}{A}\left(1 - \frac{x_1}{L}\right) I_3^2 \\
& + 2(I_3 - I_1)^2\left[\frac{\rho L}{A}\left(1 - \frac{x_1}{L}\right) I_3 - \alpha_{np} T_h\right] \\
& + 2(I_3 - I_1)^2\Big[I_3^2\,\frac{\rho L}{A}\left(1 - \frac{x_1}{L}\right) + I_1^2\,\frac{\rho L}{A}\left(1 - \frac{x_1}{L}\right) \\
& + 2\,\frac{KA}{L}\,\frac{T_c}{1 - \frac{x_1}{L}} + \frac{2KA}{L}\left(\frac{L}{x_1}\right) T_h\Big] \\
& \div \left[\alpha_{np}(I_1 - I_3) + 2\,\frac{AL}{x_1(L - x_1)}\right]\Bigg\}\,. \qquad (26)
\end{aligned}
$$

To arrive at equation (26) it was necessary to assume that the resistivity and thermal conductivity are the same for both the p and n elements, that no contact resistance occurs at any of the junctions, and that the

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*See Appendix 5.

elements are the same size. Now it is seen that the C.O.P. is a function of $\frac{L}{A}$, $\frac{x_1}{L}$, I_1 and I_3.

It would be extremely difficult to mathematically find the extremal values of the C.O.P. for the modified cascade. It would require taking the partial derivatives with respect to each of the four variables and setting them equal to zero. Expressed mathematically,

$$\frac{\partial(\text{C.O.P.})}{\partial(L/A)} = 0 ,$$

$$\frac{\partial(\text{C.O.P.})}{\partial(x_1/L)} = 0 ,$$

$$\frac{\partial(\text{C.O.P.})}{\partial I_1} = 0 ,$$

and

$$\frac{\partial(\text{C.O.P.})}{\partial I_3} = 0 .$$

In order to find the optimal value, a program was written for the digital computer which numerically optimized equation (26). One observation drawn from the results of this program is that the C.O.P. is independent of $\frac{L}{A}$. In order to understand this, it is necessary to go back to investigate a simple thermocouple. From equation (1-4) in Appendix I it is seen that the only requirement for the geometry of a simple thermocouple operating at maximum C.O.P. is

$$\frac{L_p}{A_p}\frac{A_n}{L_n} = \sqrt{\frac{\rho_n \kappa_p}{\rho_p \kappa_n}} .$$

But it has already been assumed that the electrical resistivity and thermal conductivity are the same for both elements. Thus

$$\frac{L_p}{A_p}\frac{A_n}{L_n} = 1 ,$$

and since it was also assumed that the elements were the same size, this condition is automatically satisfied. The results of changing $\frac{L}{A}$ only effects the amount of heat removed from the cold junction.

Equation (26) was solved for the following conditions:

$$\begin{aligned} T_h &= 300°K \\ T_c &= 250°K \\ \alpha_{np} &= 424 \times 10 \text{ volts/°K} \\ \rho &= 0.001 \text{ ohm-cm} \\ k &= 0.02 \text{ watts/cm-°K} \\ L &= 1 \text{ cm} \end{aligned}$$

The relative extremal values of the C.O.P. are tabulated in Table 4 for ten different cross-sectional areas.

In general the results of Table 4 show that for any two sets of data, a and b;

$$\frac{A_a}{A_b} = \frac{I_{1a}}{I_{1b}} = \frac{I_{3a}}{I_{3b}} = \frac{Q_{c_a}}{Q_{c_b}} .$$

The slight discrepancies in the table arise from the fact that the numerical optimization is not exact. The variables x_1, I_1 and I_3 were altered by a finite amount. In this case $\Delta x_1 = 0.005$, $\Delta I_1 = 0.1$ and

Table 4. Maximum C.O.P. for Various Cross-Sectional Areas

$A(cm^2)$	C.O.P.	X_1(cm)	I_1(amps)	I_3(amps)	Q_c(watts)	T_3 (°K)
.3300	.220	.600	15.9	7.36	.313	263
.3320	.220	.580	16.3	7.82	.323	264
.3540	.220	.585	17.2	8.26	.339	263
.3600	.220	.585	17.5	8.40	.345	263
.3700	.221	.590	18.0	8.64	.356	263
.3880	.220	.590	18.8	9.40	.375	263
.4200	.221	.595	20.4	9.97	.404	263
.5260	.221	.595	27.3	13.1	.540	263
1.1280	.221	.590	55.0	26.4	1.089	263
1.7720	.221	.590	86.4	41.5	1.711	263

$\Delta I_3 = 0.02\ I_1$. Thus the answers for the optimal C.O.P. are only correct to within this accuracy.

In order to establish some criteria upon which to judge this modified cascade, consider the following two possibilities:

(1) A single-stage thermocouple made of the same material and with elements of length 1 cm. and cross-sectional area of 0.370 cm^2.

(2) A two-stage cascade of the same material and with the same cooling capacity as the single-stage.

For the single-stage thermocouple the following performance is obtained;

$$C.O.P._{max} = 0.158$$

$$Q_c = 0.227 \text{ watts}$$

And for the conventional two-stage thermocouple the results are:

$$C.O.P._{stage\ 1} = 0.717$$

$$C.O.P._{stage\ 2} = 0.920$$

$$C.O.P. = 0.251$$

$$Q_{c\ stage\ 1} = 0.227 \text{ watts}$$

$$Q_{c\ stage\ 2} = 0.534 \text{ watts}$$

Since the modified cascade has a C.O.P. of about 0.22, there results an increase of about thirty-nine per cent in the C.O.P. as compared to a single-stage thermocouple. Also the modified cascade removes over fifty-two per cent more heat at its optimal C.O.P. than does a single-stage thermocouple under similar conditions.

The conventional two-stage thermocouple operates with a C.O.P. of about fourteen per cent higher than the modified cascade. However, if the conventional two-stage cascade had the same cooling capacity it would be necessary that its second stage remove 2.4 times this amount.

Thus the modified cascade operates nearly as efficiently as a standard two-stage cascade without the requirement of a large amount of heat removal in the second stage which necessitates more material and fabrication time.

In order to see the effect of the variables x_1, I_1, and I_3 on the C.O.P. of the modified cascade, the results are shown graphically on Figures 9, 10, and 11. Figure 9 shows C.O.P. vs. x_1 for various

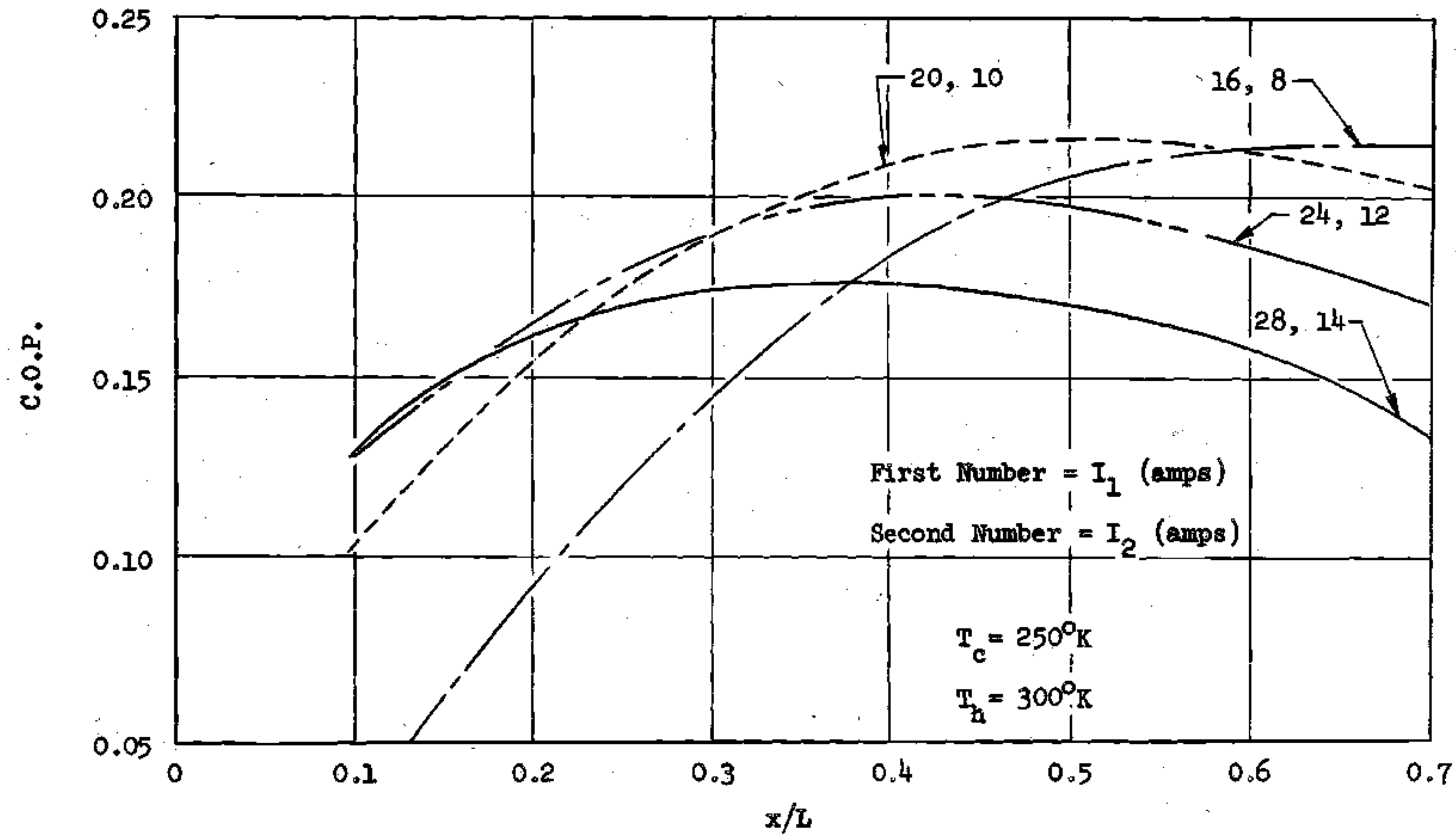

Figure 9. C.O.P. vs. Center Tap Distance from the Hot Junction for Various Current Combinations.

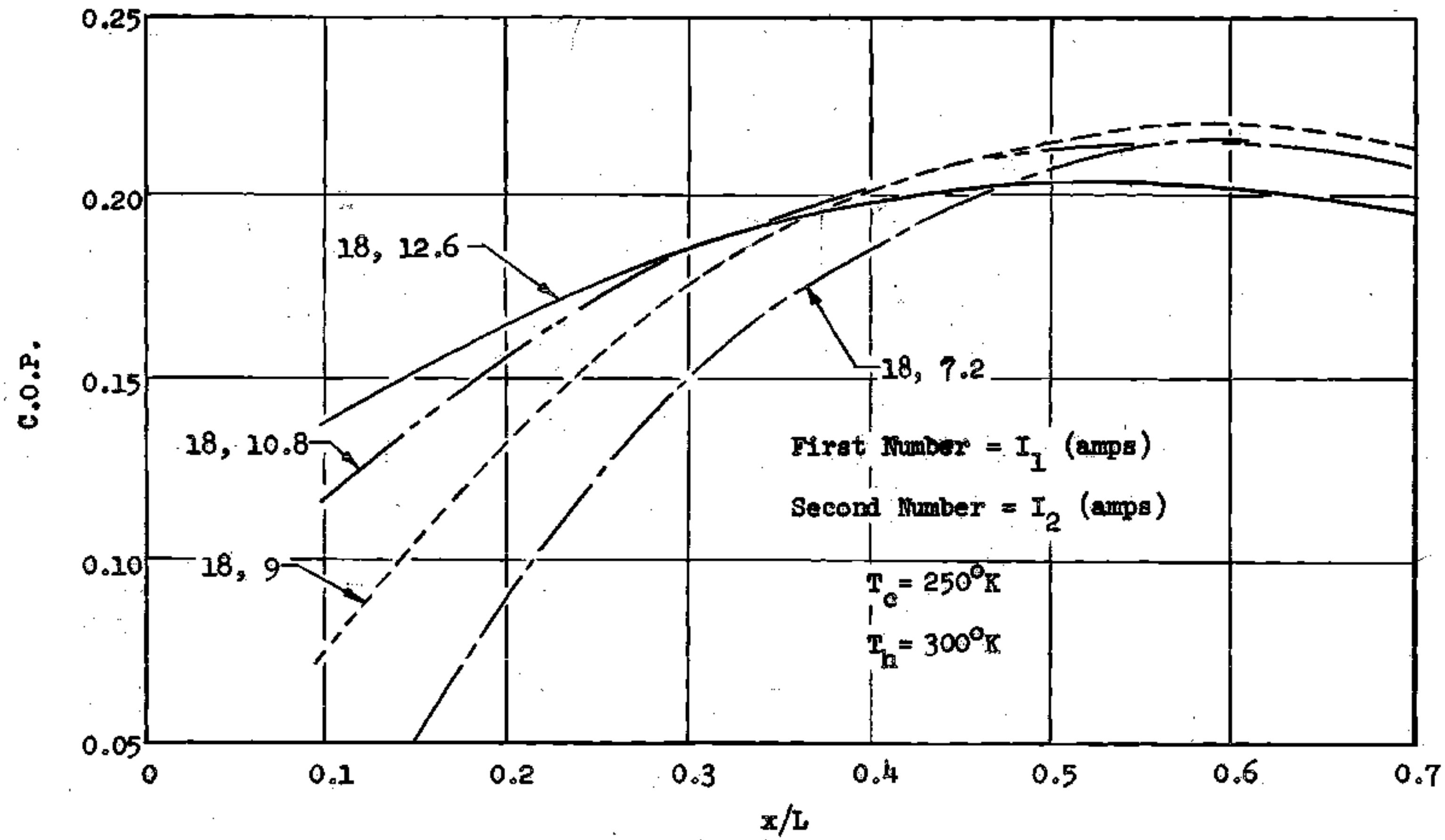

Figure 10. C.O.P. vs. Center Tap Distance from the Hot Junction for Various Current Combinations.

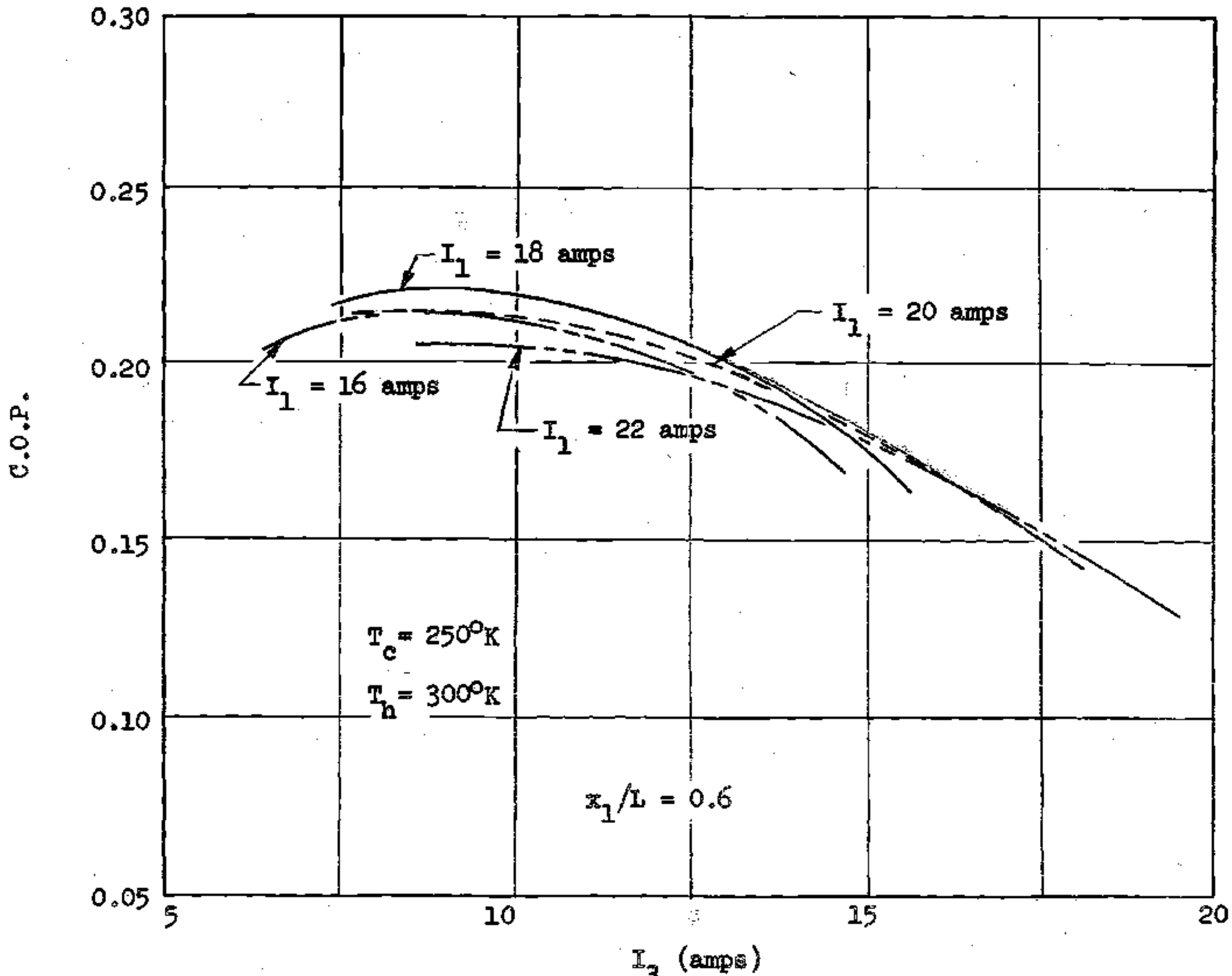

Figure 11. C.O.P. vs. Cold Junction Current.

combinations of I_1 and I_3 where I_1 equals 2 I_3. Figure 10 shows C.O.P. vs. x_1 for various combinations of I_1 and I_3 where I_1 is fixed at 18 amps and I_3 is varied. Figure 11 shows C.O.P. vs. I_3 for various values of I_1. The following observations may be made from these results:

(1) As the input current increases the optimal position for the center tap is shifted toward the hot junction. (This reduces Joule heating near the cold junction.)

(2) The optimal relationship between I_1 and I_3 is approximately given by $I_3 = 2I_1$.

Heat Transfer Analysis

In the comparison between the modified cascade thermocouple and the single stage thermocouple, one observation from the numerical example is of particular interest. Although the single stage thermocouple removes seventy-two per cent more heat due to the Peltier cooling than the modified cascade couple, the net heat removal of the modified couple is fifty-two per cent greater. In order to see the mechanics of this increased cooling a study will be made of the energy transfer in the thermocouple legs and the cold junction.

For the basis of comparison consider the following:

(1) A modified cascade thermocouple in which:

$$I_2 = 11 \text{ amps}$$

$$I_1 = 22 \text{ amps}$$

$$x_1 = 0.5 \text{ cm}$$

$$L = 1.0 \text{ cm}$$

$$A = 0.370 \text{ cm}^2$$

$$\alpha_{np} = 424 \times 10^{-6} \text{ volts/°K}$$

$$T_h = 300°K$$

$$T_c = 250°K$$

$$\rho = 10^{-3} \text{ ohm-cm}$$

$$\mathcal{K} = 0.02 \text{ watts/cm-°K}$$

(2) A single-stage thermocouple in which:

$I = 22$ amps (The same as input to modified cascade)

or,

$I = 14.42$ amps (The optimal single-stage current)

$L = 1.0$ cm

$A = 0.370 \text{ cm}^2$

$\alpha_{np} = 424 \times 10^{-6}$ volts/°K

$T_h = 300°K$

$T_c = 250°K$

$\rho = 10^{-3}$ ohm-cm

$\mathcal{K} = 0.02$ watts/cm°K

When the heat at the cold junction is broken down into its component parts, the results in Table 5 are obtained.* The modified cascade operates with

*See Appendix 6 for details.

a C.O.P. over thirty-one per cent higher than the simple thermocouple, and at this level of performance removes 110 per cent more heat. If the

Table 5. Cold Junction Analysis

I(amps)	Conduction Heat (Watts)	Joule Heat (Watts)	Peltier Heat (Watts)	Q (Watts)	Work (Watts)	C.O.P.
Simple Thermocouple						
22.00	+0.740	+1.34	-2.34	-0.287	3.08	0.093
11.00	+0.740	+0.337	-1.19	-0.090	0.866	0.104
14.41	+0.740	+0.560	-1.53	-0.227	1.42	0.158
Modified Cascade Thermocouple						
	+0.525	+0.164	-1.17	-0.479	2.28	0.211

conduction equation is solved treating the Joule heating as a homogeneous heat generation source, it is possible to find the temperature as a function of distance along the thermocouple legs. Then the heat deposited at the cold junction simply becomes the gradient of the temperature at that point multiplied by the conductivity of the leg. Thus it is highly desirable to minimize this slope. The plots of temperature vs. x for three cases are shown in Figure 12.

Variable Cross-Sectional Area

Another attempt to improve the C.O.P. of the modified cascade was to have a step discontinuity in the cross-sectional area of each leg in the location of the center tap. That is to have one cross-sectional area

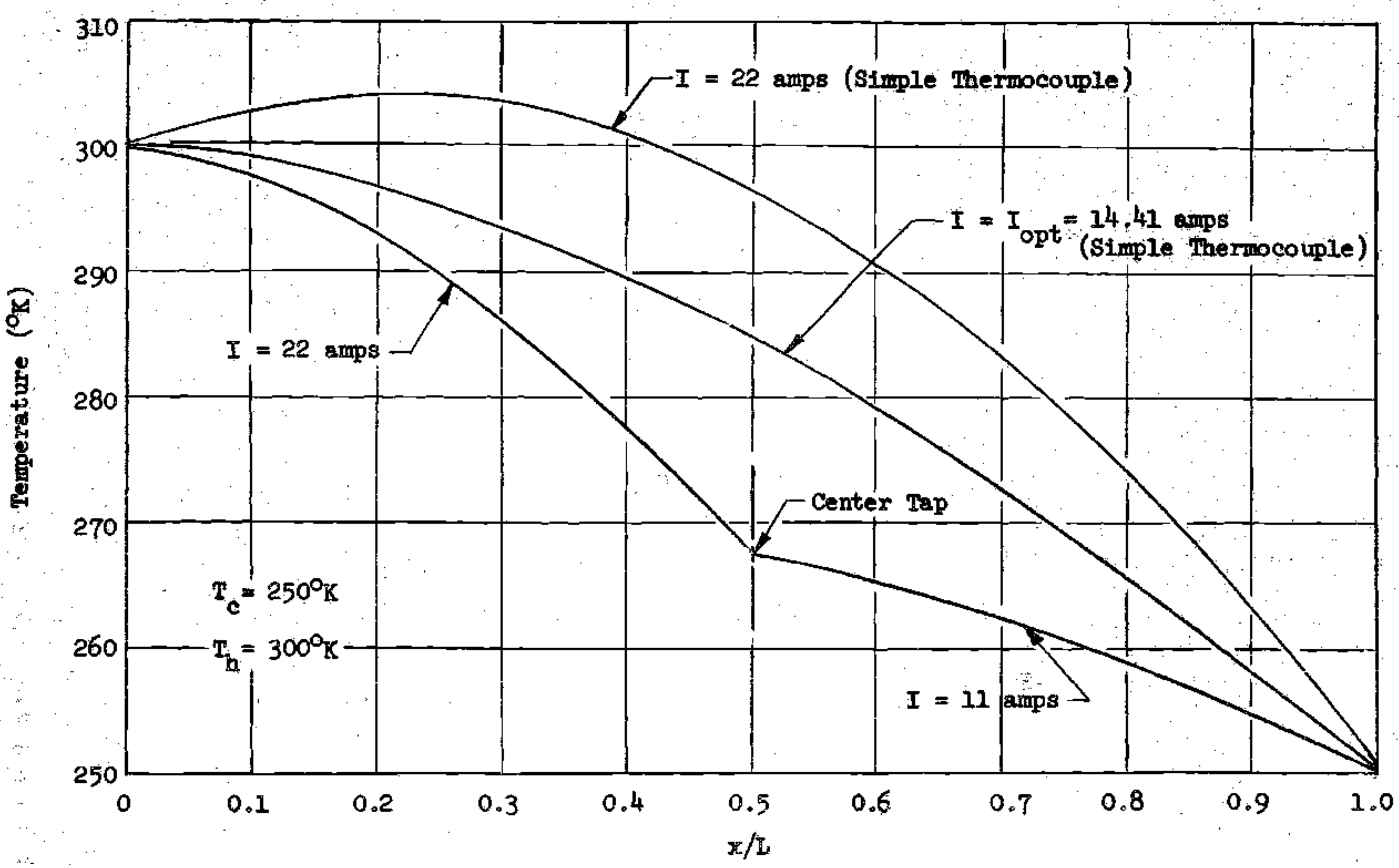

Figure 12. Temperature Gradient in Leg of Thermocouple.

for the portion of the legs between the center tap and the hot junction and to have a different cross-sectional area for the remaining portion of the legs. The same thermocouple materials and source and sink temperatures were used as in the previous analyses. The cross-sectional area between the hot junction and center tap was held constant at 0.370 cm^2 while x_1, I_1, I_3 and the cross-sectional area in the neighborhood of the cold junction were varied. The results for some of the relative maxima are shown in Table 6. This reflects the information from Table 4 inasmuch as the geometry of the thermocouple does not seem to affect its C.O.P., but only its cooling capacity. The table also shows that as the cross-sectional area near the cold junction increases, the optimal location for the center tap moves away from it. (This is to keep the thermal resistance from being reduced too drastically and to inhibit heat conduction to the cold junction.)

Table 6. Maximum Values of C.O.P. for Modified Cascade with Two Different Cross-Sectional Areas.

(A = 0.370 cm at hot junction)

Cold Junction Area (cm^2)	C.O.P.	x_1(cm)	I_1(amps)	I_3(amps)	Q (watts)	T (°K)
0.302	.2205	.635	16.7	8.02	.33	263
0.348	.2205	.610	17.5	8.40	.345	263
0.386	.2205	.590	18.2	8.74	.360	263
0.426	.2205	.565	19.0	9.12	.376	263
0.466	.2205	.540	19.8	9.50	.391	263
0.508	.2205	.520	20.6	9.89	.407	263
0.548	.2205	.500	21.4	10.3	.423	263
0.586	.2205	.485	22.1	10.6	.43	263

Heat Removal Comparison

As the next basis of comparison consider the maximum cooling capacities of the modified cascade and a simple single-stage thermocouple. The cooling of a simple thermocouple was given by equation (10) as

$$Q_c = \alpha_{np} T_c I - \frac{1}{2} I^2 R - K\Delta T \ .$$

From this expression the optimal current is found to be

$$I_{opt} = \frac{\alpha_{np} T_c}{R} ,$$

and when this is substituted back into equation (10) the maximum cooling is

$$Q_{max} = \frac{(\alpha_{np} T_c)^2}{2R} - K(T_h - T_c) \ .$$

This expression can be rewritten, when the definitions of R and K are used, in the following manner;

$$Q_{max} = \frac{A}{L}\left[\frac{(\alpha_{np} T_c)^2}{4\rho} - 4K(T_h - T_c)\right] \ . \qquad (27)$$

From equation (27) it can be seen that theoretically Q_{max} is unbounded. Thus to get a value for Q_{max} it is necessary to prescribe a geometry of the thermocouple. This argument can also be applied to the modified cascade with the same results. That is, in theory Q_{max} approaches infinity as:

a) $\frac{A}{x_1} \rightarrow \infty$

b) $\frac{A}{L - x_1} \rightarrow \infty$

c) $\frac{A}{L} \rightarrow \infty$.

So to get some idea of relative values consider the following geometries:

(1) For the simple thermocouple

$$A = 0.370 \text{ cm}^2$$

$$L = 1 \text{ cm}$$

$$\alpha_{np} = 424 \times 10^{-6} \text{ volts/}^{\circ}\text{K}$$

$$\rho = 10^{-3} \text{ ohm-cm}$$

$$\mathcal{K} = 0.02 \text{ watts/cm-}^{\circ}\text{K}$$

(2) For the modified cascade the same values apply. Substitution into equation (27) shows that for the simple thermocouple the following values are obtained:

$$Q_{max} = 0.30 \text{ watts}$$

$$I_{opt} = 19.63 \text{ amps}$$

From equation (15) it is ascertained that the C.O.P. at Q_{max} is 0.1225.

For the modified cascade it is necessary to specify a value for x_1 before determining Q_{max}. The results for various values of x_1 are given in Table 7.

Thus it is seen that the modified cascade is capable of removing over three times as much heat as a simple thermocouple of the same size.

Table 7. Cooling Capacity of Modified Cascade for Various Center Tap Locations

x_1(cm)	Q_{max}(watts)	I_1(amps)	I_3(amps)	C.O.P.
0.1	1.14	172	20.7	0.034
0.2	1.05	101	18.3	0.053
0.3	0.943	68.0	17.7	0.074
0.4	0.84	51.9	16.6	0.091
0.5	0.739	42.1	15.2	0.104
0.6	0.639	36.6	14.6	0.110
0.7	0.539	30.0	15.0	0.118
0.8	0.456	26.0	13.0	0.125

Maximum Temperature Difference

Another basis of comparison between the modified cascade thermocouple and a simple thermocouple is the maximum temperature difference across which they can operate. For a simple thermocouple this was expressed by equation (19) as

$$\Delta T_{max} = 2T_m \frac{\sqrt{1 + ZT_m} - 1}{\sqrt{1 + ZT_m} + 1} .$$

Note that, as in the case of the C.O.P., the geometry of the simple thermocouple does not affect the value of ΔT_{max}. The expression which gives the maximum sink temperature for any given source temperature for a modified cascade is*

*See Appendix 7 for development.

$$T_h = \frac{(\alpha_{np} I_2 + 2K_2 + 2K_1)(I_3 \alpha_{np} T_c + 2K_2 T_c - I_3^2 R_2)}{4K_1 K_2} - \frac{I_3^2 R_2 + I_1^2 R_1 + 2K_2 T_c}{2K_1} \text{ . } \quad (28)$$

Notice that while equation (19) looks simple, it requires a trial and error solution. This is because T_m cannot be evaluated until both T_h and T_c are known. Needless to say, an exact solution to the optimal value of T_h from equation (28) would be extremely difficult. Again, in order to get a basis of comparison, a specific example will be used. Assume that the following conditions prevail for both thermocouples:

$$T_c = 250°K$$
$$\alpha_{np} = 424 \times 10^{-6} \text{ volts/°K}$$
$$\rho = 10^{-3} \text{ ohm-cm}$$
$$\kappa = 0.02 \text{ watts/cm-°K}$$
$$L = 1 \text{ cm} .$$

From equation (16a) it can be verified that

$$Z = 2.245 \times 10^{-3} \frac{1}{°K} .$$

Thus for the simple thermocouple,

$$\Delta T_{max} = 70.4°K .$$

To obtain solutions to equation (28) for the modified cascade, a computer solution was used which numerically optimized the solution. The results are shown in Table 8. Notice that the location of the center tap is the same for the various cross-sectional areas, and T_h is essentially the same. Also, notice that I_1 is always equal to 10 I_3. Another point of interest is that $T_{h_{max}}$ would be approximately 435°K for a standard two stage-cascade. Thus the modified cascade comes close to duplicating the maximum temperature difference of a standard cascade thermocouple.

Table 8. Maximum Hot Junction Temperature for a Modified Cascade with Cold Junction Temperature of 250°K.

x_1 (cm)	A (cm^2)	I (amps)	I (amps)	T (°K)
0.115	0.064	35.3	3.53	414
0.115	0.094	51.2	5.12	414
0.115	0.122	66.3	6.63	414
0.115	0.150	81.0	8.10	414
0.115	0.198	108.3	10.8	414
0.115	0.222	120	12.0	414
0.115	0.370	201	20.1	414

CHAPTER IV

EXPERIMENTAL RESULTS

In order to check the theory of the modified cascade heat pump, an experimental model was built. The semiconductor elements were a bismuth telluride product called melcor,* and were one quarter of an inch long and one quarter of an inch in diameter. These elements were cut to size and tinned through the courtesy of Borg-Warner Corporation. A schematic representation of the experimental apparatus is shown in Figure 13. After the thermocouple was fabricated, it was completely covered with a type of insulation known as 'Santocel "A" '** which was donated by the Monsanto Chemical Company.

The input power was supplied by a Sorenson Nobatron DCR-20-125 power supply (0-25 volts, 0-150 amps). The currents, I_1 and I_2, were measured by using a potentiometer to measure the voltage drop across the shunts inserted in the circuit. Copper-constantan thermocouples were made of 36 gage wire and were used to measure the hot and cold junction temperatures and the center tap temperature. In each case the thermocouple was imbedded in the copper plate at the junction as close as possible to the center of the semiconductor element. The cold junction heater was fabricated by wrapping nichrome wire in a slot around the edges of the copper conductor which formed the cold junction.

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*See Appendix 9 for property data.

**See Appendix 10 for property data.

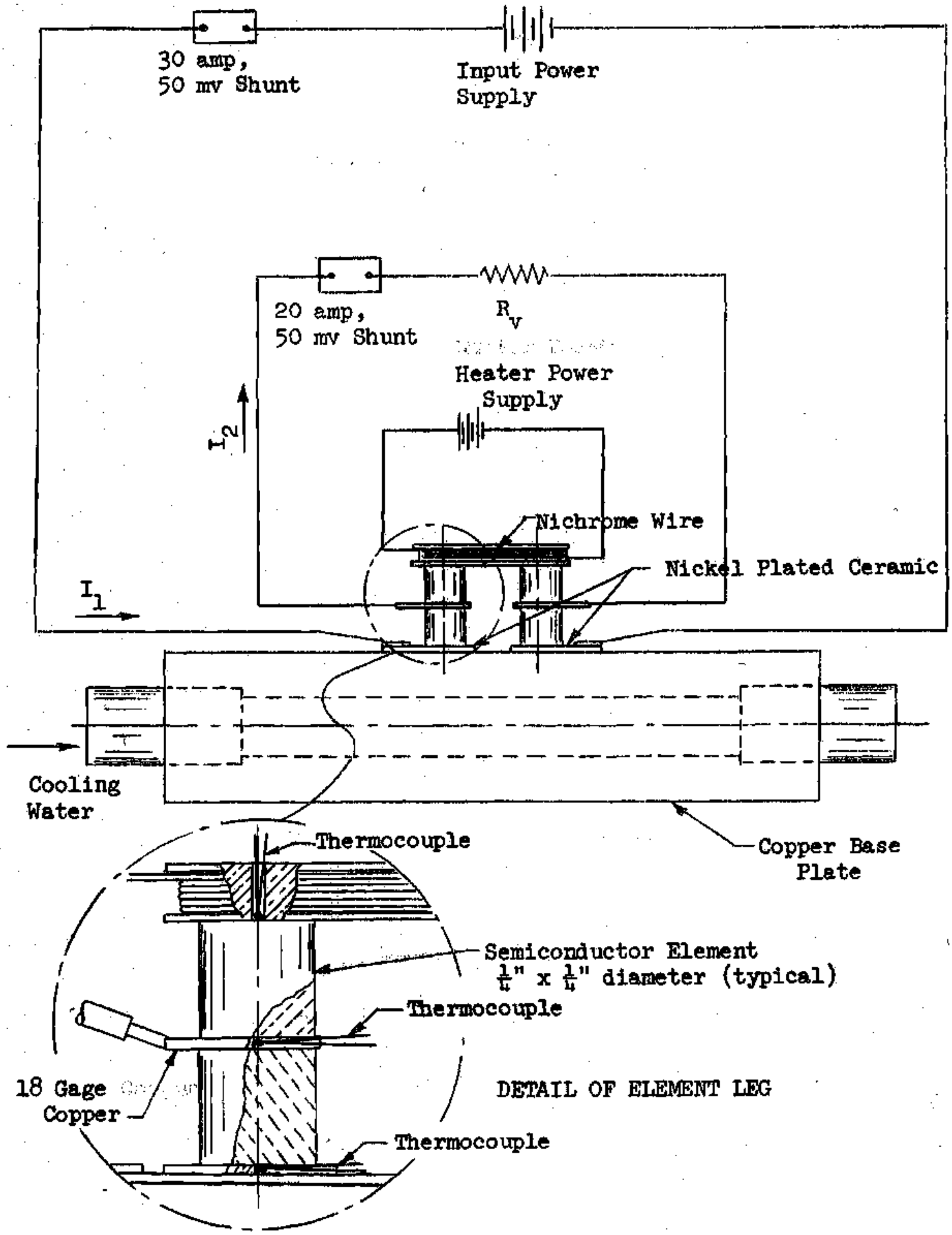

Figure 13. Schematic of Experimental Apparatus.

Since the elements came tinned with a bismuth solder that melts at approximately 250°F, it was necessary to use solders in the fabrication that melt at a lower temperature so that the tin on the element would not be damaged. In order to accomplish this, two types of solders were mixed. One of these was Wood's Metal (which is by weight 15 parts bismuth, 8 parts lead, 4 parts cadmium and 4 parts tin) which melts at about 150°F. The second type of solder was composed of 32 per cent lead, 15.5 per cent tin and 52.5 per cent bismuth by weight and melted at about 205°F. It was necessary to have the two solders with different melting points so that when soldering successive junctions on the same element, it was possible to proceed from the solder with the highest melting point to that with the lowest melting point. This enables the second junction to be soldered without remelting the first junction.

In mounting the elements on the base plate, it was necessary that the two legs be electrically insulated, but in good thermal contact with the base plate. This was accomplished by using two thin (1 mm thick) pieces of ceramic, plated on both sides with nickel, which were also donated by Borg-Warner Corporation. These plates were first soldered to the base plate and then one leg of the element was soldered to each of them.

When the first experimental values for the C.O.P. were found, a large discrepancy existed between them and the analytical values. This was largely resolved when the electrical contact resistances were considered. These were neglected in the analytical treatment since they are usually quite small and difficult to predict. In this case they were not negligible due to the fact that there were two different types of solder at each junction, and also due to a high degree of ineptness in the art of fabrication.

In order to measure the contact resistances, the center tap of the thermocouple was left open and a small, carefully measured current was passed through the couple. Since it is difficult to measure resistance, the voltage drops were measured at various points with a potentiometer and the resistances were then found from Ohm's law. In order to eliminate any back emf caused by the Seebeck effect, heat was added at the cold junction so that it remained at the same temperature as the hot junction and the potentiometer terminals. Details of the procedure of determining this resistance are given in Appendix 10.

Experimental runs were made to determine the performance of the thermocouple used as a modified cascade, and also as a simple thermocouple (with the center tap open). The data from these runs was used to evaluate the coefficient of performance, the cooling capacity and the maximum temperature difference of the two types of thermocouples. It was not possible to determine exact analytical values because the tolerances in the measurements for the property values were large. It was necessary to use the two extremal sets of property values to plot the upper and lower limits of the analytical curves. This was done both with and without contact resistance considerations. Figure 14 shows the no-load performance of the modified cascade thermocouple, and Figure 15 shows the no-load performance of a simple thermocouple. The agreement of the simple thermocouple with analytical results is better than that of the modified cascade. This is due, in part, to the heat leak of the modified cascade through the center tap connection. A set of revised data

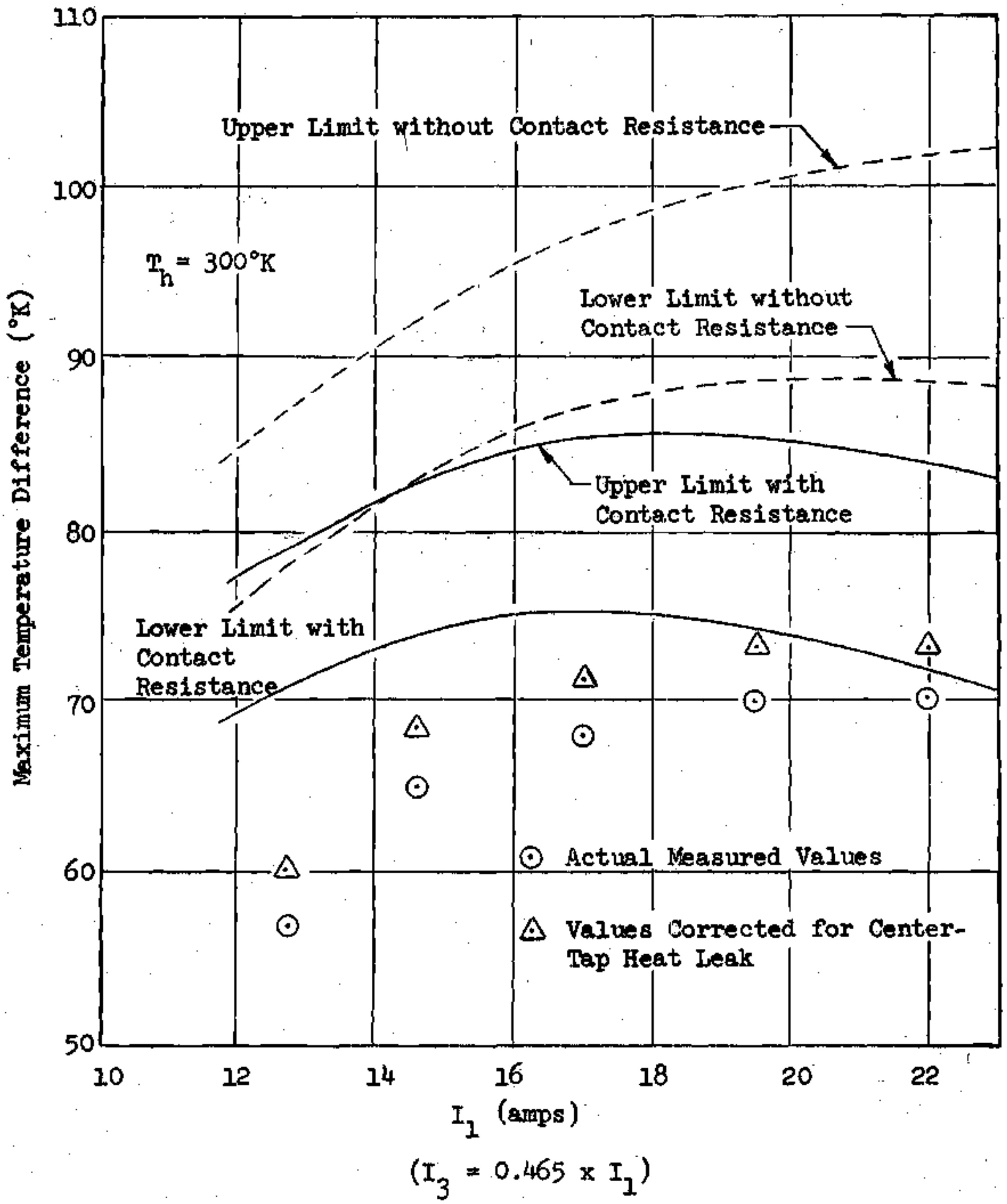

Figure 14. Modified Cascade Thermocouple Under No-Load

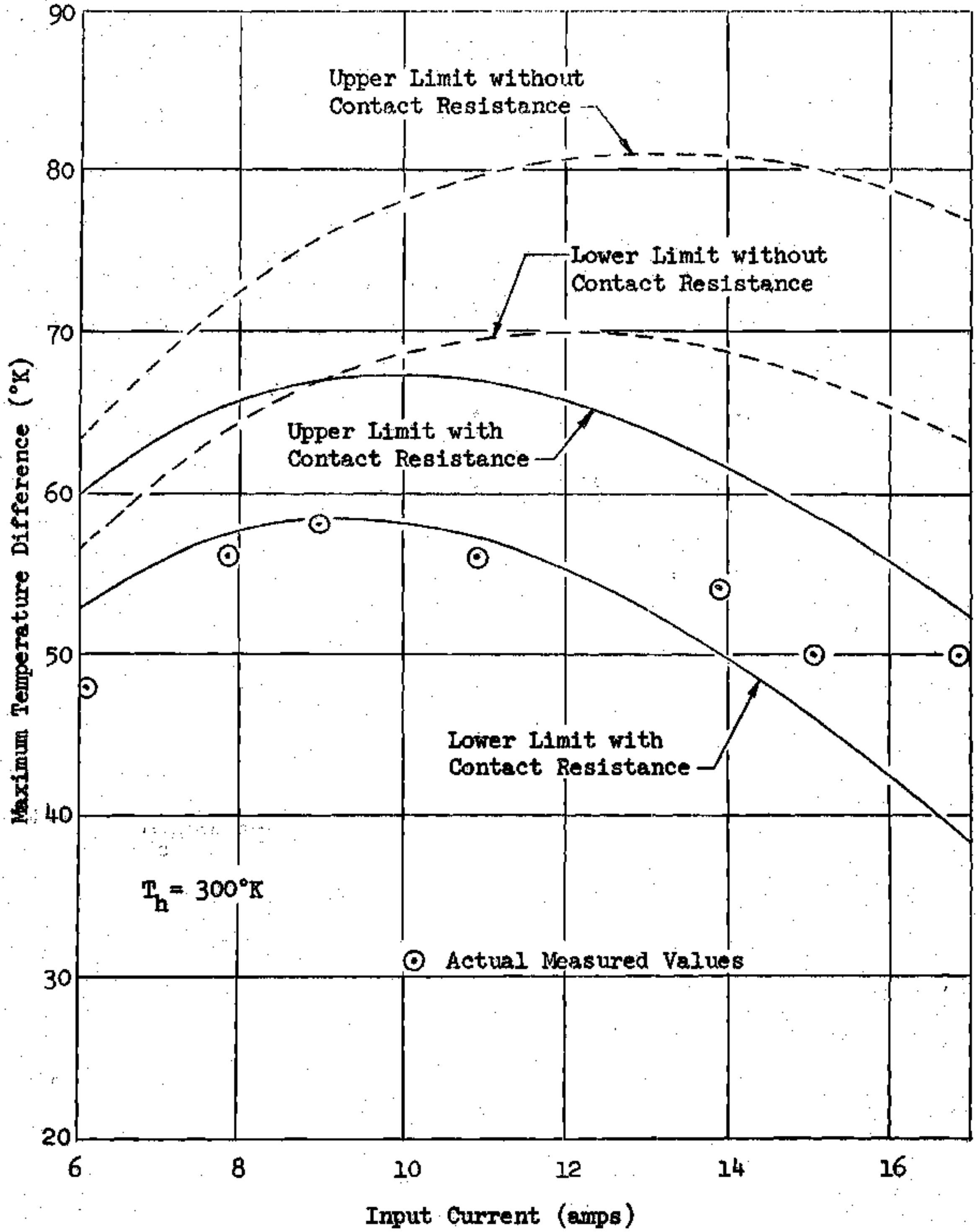

Figure 15. Simple Thermocouple under No-Load.

points compensating for this heat leak showed an improved correlation. This heat leak was not a problem with the simple thermocouple, since it was possible to insulate the center tap leads when they were not connected. Another factor accounting for the discrepancy between the analytical and experimental results is that the properties of the elements were assumed to be independent of temperature and equal to the values measured at 300°K. Especially significant is the fact that the Seebeck coefficient decreases with decreasing temperature. Finally, any heat leak through the insulation, even of the order of 0.1 watt, would cause an appreciable error. The maximum measured temperature difference with the modified cascade heat pump was about 74°K. For the simple thermocouple it was 58°K.

Figures 16 and 17 show the coefficient of performance for the modified cascade and the simple thermocouple respectively. Again, the performance of the simple thermocouple lies closer to the analytical values than does that of the modified cascade thermocouple until the center tap heat leak is considered. The coefficient of performance of the modified cascade is roughly twice that of the simple thermocouple. It should be pointed out that the simple thermocouple is penalized by the contact resistances at the center tap; generally simple thermocouples are made without center taps.

Figure 18 shows the heat removal at the cold junction for the modified cascade thermocouple and Figure 19 shows the same thing for the simple thermocouple. The modified cascade removes approximately three times as much heat at maximum cooling capacity as does the simple thermocouple. Also the modified cascade thermocouple has a coefficient

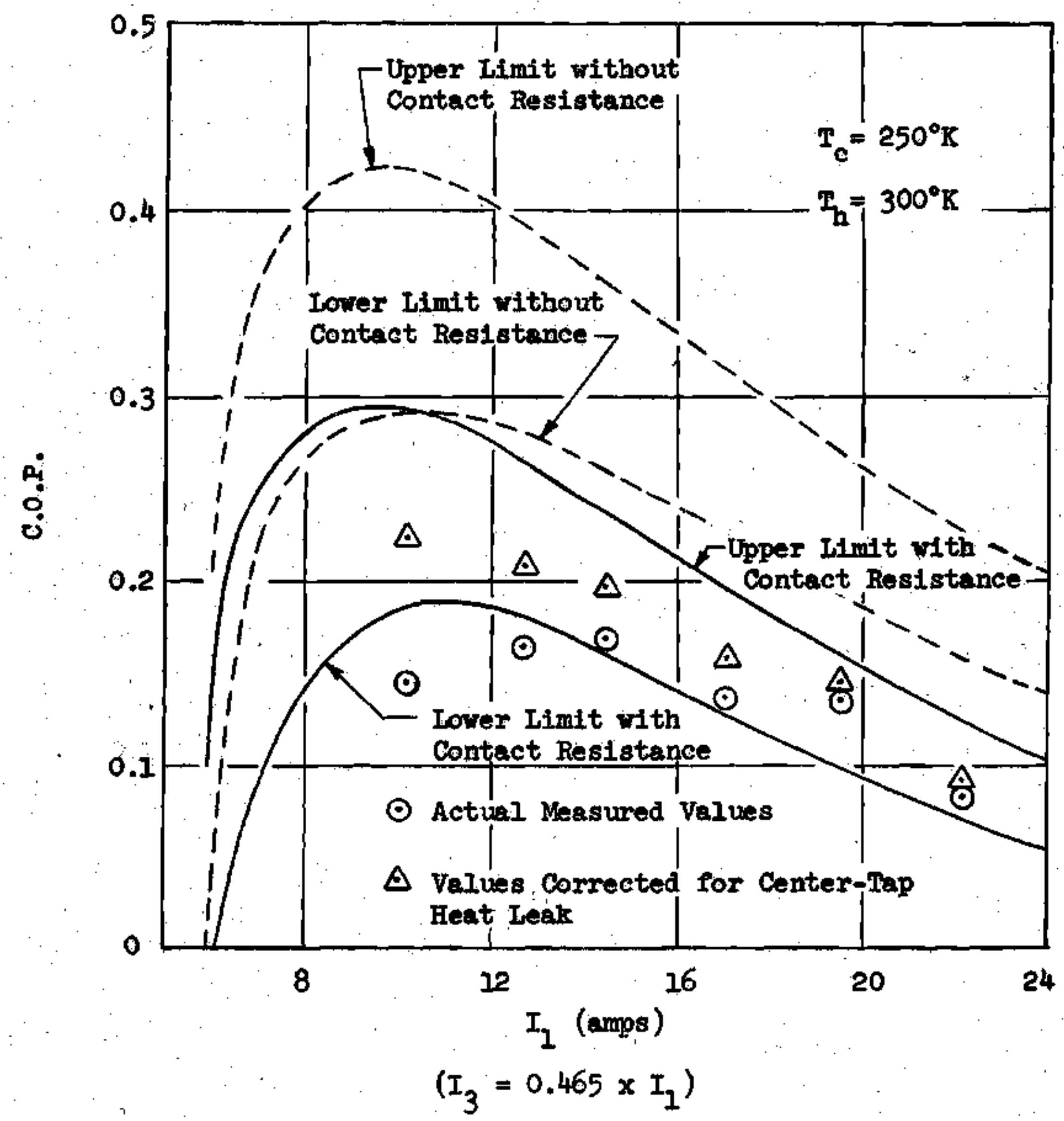

Figure 16. Coefficient of Performance for a Modified Cascade Thermocouple as a Function of Current.

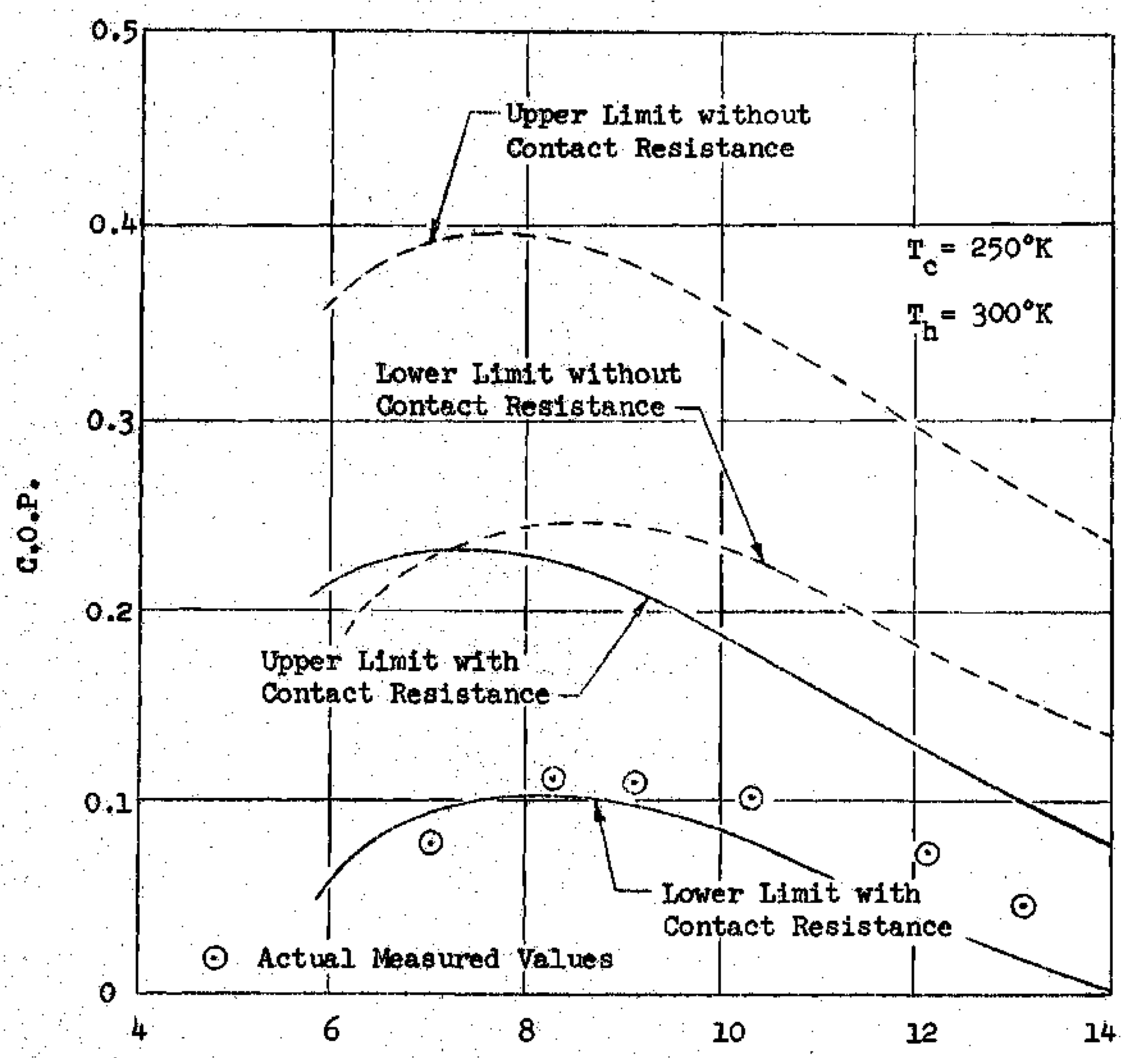

Figure 17. Coefficient of Performance for a Simple Thermocouple as a Function of Current.

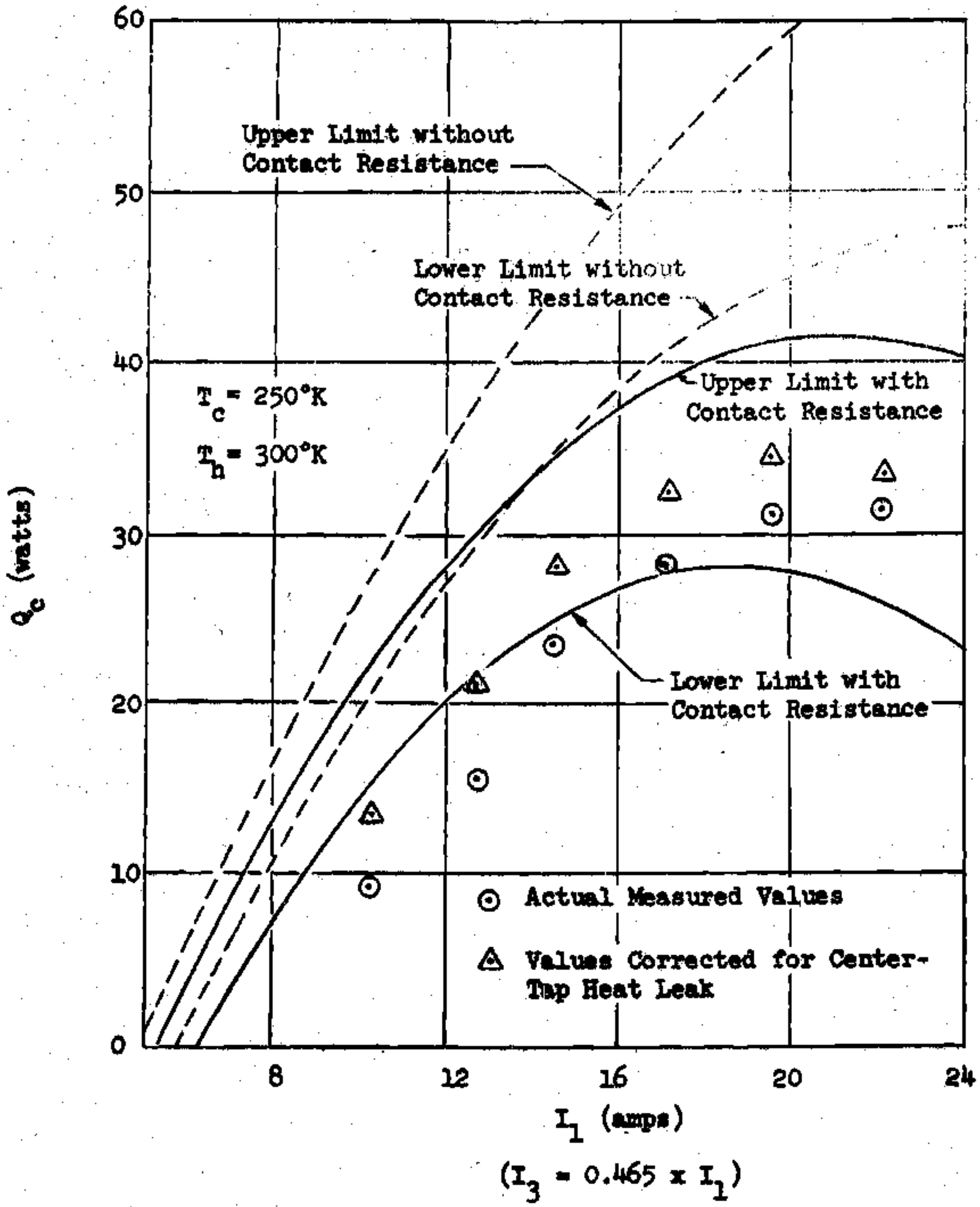

Figure 18. Cold Junction Heat Removal for a Modified Cascade Thermocouple as a Function of Current.

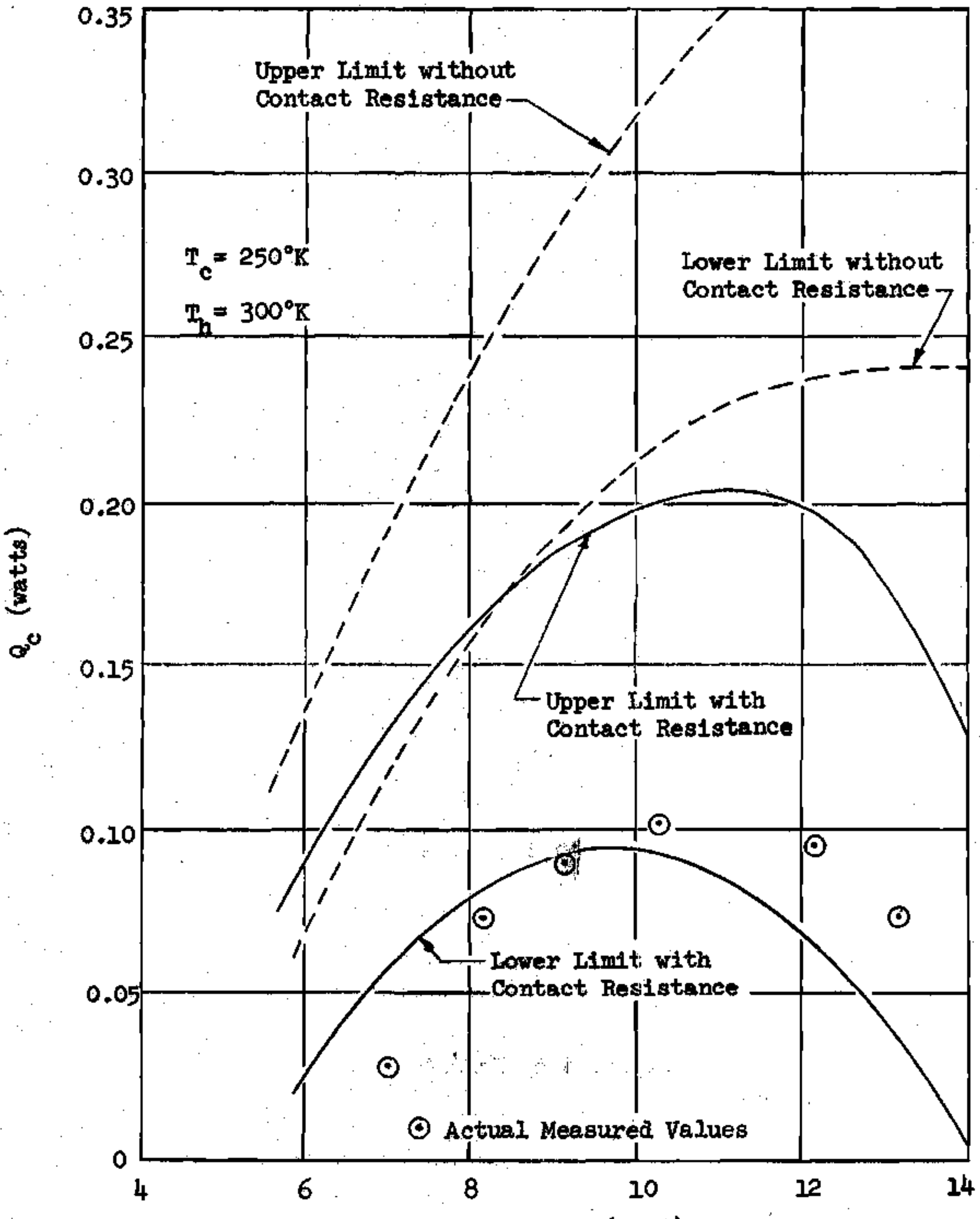

Figure 19. Cold Junction Heat Removal for a Simple Thermocouple as a Function of Current.

of performance which is about 50 per cent higher at this peak cooling load than the simple thermocouple under its peak cooling load.

The data obtained in the experimental work and the explanation of data treatment and analytical equations used in plotting the curves in this chapter are found in Appendix 11.

CHAPTER V

CONCLUSIONS

The major conclusions to be drawn from the experimental and analytical investigation of this work are given briefly in the following paragraphs.

Even in the ideal case, with no contact resistance, cascading offers very small gains in the maximum C.O.P. over a single-stage thermoelectric heat pump except in the case where ΔT approaches ΔT_{max} for the single-stage device. Any contact resistance in fabrication would tend to diminish the gain the cascaded thermocouple provided. This can be seen more clearly in Figure 5 and from equation (25).

In most applications of a single-stage thermoelectric heat pump, over 50 per cent of the Peltier cooling at the cold junction is used to remove the heat deposited at the cold junction due to Joule heating and conduction heat transfer from the hot junction. Thus a method of improvement would be to reduce these effects even at a loss of Peltier cooling at the cold junctions.

If Q_c is the cooling capacity of a multiple stage cascaded thermoelectric heat pump operating at maximum C.O.P., then the maximum cooling capacity of the device at any C.O.P. is only slightly greater than Q_c. This is due to the fact that each stage must remove all the energy supplied to the preceding stage in addition to the heat removed at the cold junction. And as the stage C.O.P.'s decrease, the energy supplied to each stage must increase more rapidly than the cooling capacity.

The modified cascade thermoelectric heat pump operates at a C.O.P. less than that of a two-stage device, but higher than that of a single-stage thermoelectric heat pump. For a typical example using current materials, the C.O.P. of the modified cascade heat pump is 14 per cent less than that of a two-stage heat pump and 39 per cent higher than the C.O.P. of a single-stage heat pump. The modified cascade heat pump is superior to a two-stage cascade thermocouple in the sense that its peak cooling capacity is approximately twice its cooling capacity when operating at maximum C.O.P. Thus it is able to handle large increases in the cooling load above design conditions which could not be handled by a two-stage heat pump. The modified cascade device can reach approximately the same ΔT_{max} as a two-stage device and is possibly easier to fabricate.

Building a working model of a thermoelectric heat pump which will approach the performance predicted for the ideal case is difficult. Even when the elements are already tinned, it is necessary to have at least two solders, which melt at different temperatures below the melting point of the tin on the element, to solder the junctions. Heating the element above the melting point of the tin could result in the loss of the tin. This is a real problem since tinning the semi-conductor elements is extremely difficult, and those industrial concerns which have perfected a tinning process are reluctant to divulge their methods.

APPENDICES

APPENDIX I

OPTIMIZATION OF THE COEFFICIENT OF PERFORMANCE FOR A SINGLE-STAGE THERMOCOUPLE

For a simple thermocouple of p and n elements, the coefficient of performance is expressed,

$$\text{C.O.P.} = \frac{\alpha_{np} T_c I - \frac{1}{2} I^2 R - K \Delta T}{\alpha_{np} \Delta T I + I^2 R},$$

where

$$R = \frac{L_n}{A_n} \rho_n + \frac{L_p}{A_p} \rho_p = G_n \rho_n + G_p \rho_p$$

and

$$K = \frac{A_n}{L_n} \mathcal{K}_n + \frac{A_p}{L_p} \mathcal{K}_p = \frac{\mathcal{K}_n}{G_n} + \frac{\mathcal{K}_p}{G_p}.$$

Thus

$$\text{C.O.P.} = \frac{\alpha_{np} T_c I - \frac{1}{2} I^2 [G_n \rho_n + G_p \rho_p] - \left[\frac{\mathcal{K}_n}{G_n} + \frac{\mathcal{K}_p}{G_p}\right] \Delta T}{\alpha_{np} \Delta T \; I + I^2 (G_n \rho_n + G_p \rho_p)} = \frac{N}{D}. \qquad (1\text{-}1)$$

Now for a given situation the junction temperatures, T_h and T_c, are determined, and α_{np}, ρ_n, ρ_p, $\mathcal{K}_n$ and $\mathcal{K}_p$ are also determined by the choice of material. Thus the C.O.P. becomes a function of geometry, G_n and G_p, and also the current I. In short,

$$\text{C.O.P.} = \text{C.O.P.}\ (I,\ G_p,\ G_n)$$

for most design situations. Thus at an extremal value the following relations hold:

a) $\dfrac{\partial(\text{C.O.P.})}{\partial I} = 0$

b) $\dfrac{\partial(\text{C.O.P.})}{\partial G_n} = 0$

c) $\dfrac{\partial(\text{C.O.P.})}{\partial G_p} = 0$

From (a) after rearranging is obtained,

$$0 = \alpha_{np} I^2 (T_c + \tfrac{1}{2}\,\Delta T) R - \alpha_{np} \Delta T^2 K - 2I\,\Delta T K R\ , \qquad (1\text{-}2)$$

from (b)

$$D(-\tfrac{1}{2}\, I^2\, \rho_p + \frac{\kappa_p}{G_p^{\,2}}\, \Delta T) = N(I^2 \rho_p)\ , \qquad (1\text{-}3)$$

from (c)

$$D(-\tfrac{1}{2}\, I^2 \rho_n + \frac{\kappa_n}{G_n^{\,2}}\, \Delta T) = N(I^2 \rho_n)\ , \qquad (1\text{-}4)$$

where N and D stand for the numerator and denominator of equation (1-1). Or rewriting (1-3) and (1-4);

$$\frac{\kappa_p}{\rho_p G_p^{\,2}} = \frac{I^2}{\Delta T}\, [\tfrac{N}{D} + \tfrac{1}{2}] \qquad (1\text{-}3')$$

$$\frac{\kappa_n}{\rho_n G_n^2} = \frac{I^2}{\Delta T}\left[\frac{N}{D} + \frac{1}{2}\right] \qquad (1\text{-}4')$$

Thus it can be seen that (1-3') equalt (1-4'), so mathematically

$$\frac{\kappa_p}{\rho_p G_p^2} = \frac{\kappa_n}{\rho_n G_n^2} ,$$

or

$$\frac{G_p^2}{G_n^2} = \left(\frac{L_p}{A_p}\frac{A_n}{G_n}\right)^2 = \frac{\rho_n}{\kappa_n}\frac{\kappa_p}{\rho_p} .$$

So, at optimal conditions, the geometry of the n and p elements are related by the expression,

$$\frac{L_p A_n}{L_n A_p} = \sqrt{\frac{\rho_n \kappa_p}{\rho_p \kappa_n}} . \qquad (1\text{-}5)$$

Now return to equation (1-2) to find an optimal value of the current. If $(T_c + \frac{1}{2}\Delta T)$ is defined as the mean temperature, T_m, equation (1-2) can be rewritten:

$$0 = \alpha_{np} R T_m I^2 - 2KR\Delta T I - \alpha_{np}\Delta T^2 K . \qquad (1\text{-}6)$$

Solving for I from the quadratic formula,

$$I = \frac{2KR\Delta T \pm \sqrt{4(KR)^2\Delta T^2 + 4[\alpha_{np}^2\,\Delta T^2 T_m (KR)]}}{2\alpha_{np} R T_m} . \qquad (1\text{-}7)$$

The negative sign in the preceding equation yields a negative value of I, thus only the positive sign will be considered. The quantity (KR) can be expressed,

$$KR = \left[\frac{\kappa_n}{G_n} + \frac{\kappa_p}{G_p}\right](G_n\rho_n + G_p\rho_p) = \kappa_n\rho_n + \kappa_p\rho_p + \frac{G_n}{G_p}\rho_n\kappa_p + \frac{G_p}{G_n}\rho_p\kappa_n .$$

But at optimal conditions from equation (1-5)

$$\frac{G_p}{G_n} = \sqrt{\frac{\rho_n\kappa_p}{\rho_p\kappa_n}} .$$

So that

$$KR = \kappa_n\rho_n + \kappa_p\rho_p + \sqrt{\frac{\rho_p\kappa_n}{\rho_n\kappa_p}}\,\rho_n\kappa_p + \sqrt{\frac{\rho_n\kappa_p}{\rho_p\kappa_n}}\,\rho_p\kappa_n ,$$

which simplifies to

$$KR = (\kappa_n\rho_n + 2\sqrt{\rho_n\kappa_n\rho_p\kappa_p} + \rho_p\kappa_p) = [(\rho_p\kappa_p)^{1/2} + (\rho_n\kappa_n)^{1/2}]^2 .$$

At this point define a parameter known as the figure of merit, Z, as follows:

$$Z = \frac{\alpha_{np}^2}{KR} = \frac{\alpha_{np}^2}{[(\rho_{p\ p})^{1/2} + (\rho_{n\ n})^{1/2}]^2} . \qquad (1\text{-}8)$$

With this definition equation (1-7) simplifies to

$$I = \frac{2\,\frac{\alpha_{np}^2}{Z}\,\Delta T + 2\Delta T\sqrt{\frac{\alpha_{np}^4}{Z^2} + \alpha_{np}^2\ T_m\,\frac{\alpha_{np}^2}{Z}}}{2\alpha_{np}RT_m}$$

or more simply,

$$I = \frac{\alpha_{np}\Delta T\,[1 + \sqrt{1 + ZT_m}\,]}{RZT_m} \quad . \tag{1-9}$$

Rearranging yields

$$(IR)_{Opt} = \frac{\alpha_{np}\ \Delta T}{\sqrt{1 + ZT_m} - 1} \tag{1-10}$$

Now recall equation (1-1) for C.O.P. and multiply numerator and denominator by R and obtain,

$$C.O.P. = \frac{\alpha_{np}T_c(IR) - \frac{1}{2}(IR)^2 - \Delta T(KR)}{\alpha_{np}\Delta T(IR) + (IR)^2} \quad .$$

Into the preceding expression substitute the values found in equations (1-8) and (1-10). After simplification the following expression will be obtained;

$$C.O.P._{max} = \frac{\frac{T_m}{A - 1} - \frac{T_m}{ZT_m}}{\frac{\Delta T}{A - 1} + \frac{\Delta T}{(A-1)^2}} - \frac{1}{2}$$

where

$$A = \sqrt{1 + ZT_m} \quad .$$

Clearing fractions yields

$$\text{C.O.P.}_{max} = \frac{T_m}{\Delta T}\frac{\left[ZT_m - (A-1)\right](A-1)}{AZT_m} - \frac{1}{2} = \frac{T_m}{\Delta T}\frac{\left[ZT_m - (A-1)\right](A^2-1)}{AZT_m(A+1)} - \frac{1}{2} .$$

However

$$A^2 - 1 = ZT_m$$

and the expression becomes

$$\text{C.O.P.}_{max} = \frac{T_m}{\Delta T}\frac{\left[A^2 - 1 - A + 1\right]ZT_m}{AZT_m(A+1)} - \frac{1}{2} = \frac{T_m}{\Delta T}\frac{A-1}{A+1} - \frac{1}{2} .$$

or in final form

$$\text{C.O.P.}_{max} = \frac{T_m}{\Delta T}\frac{\sqrt{1 + ZT_m} - 1}{\sqrt{1 + ZT_m} + 1} - \frac{1}{2} . \qquad (1\text{-}11)$$

Equation (1-11) yields the maximum attainable C.O.P. for any thermocouple operating between two fixed temperatures of T_h and T_c. It is a function of only one parameter, the figure of merit, Z.

APPENDIX 2

NUMERICAL OPTIMIZATION OF THE C.O.P. OF A CASCADE SYSTEM

A digital computer program was written to determine the optimum C.O.P. for a cascade system of n stages. The variables were T_2, $T_3, \ldots T_{n-2}, T_{n-1}$. T_1 was the source temperature and T_n was the sink temperature, and these values remained constant. To begin the program, arbitrary values for the interstage temperatures were assumed and an initial C.O.P. was computed from the equation

$$\text{C.O.P.} = \text{C.O.P.}(T_2, T_3, \ldots, T_{n-1}) \ .$$

Next T_2 was replaced by $T_2 + \Delta T_2$ and a new value of the C.O.P. was found,

$$\text{C.O.P.}_2 = \text{C.O.P.}(T_2 + \Delta T_2,\ T_3, \ldots T_{n-1}) \ .$$

If $\text{C.O.P.}_2 > \text{C.O.P.}$ then changing T_2 to $T_2 + \Delta T_2$ was apparently in the right direction and the program proceeded to change T_3 to $T_3 + \Delta T_3$. However, if $\text{C.O.P.}_2 < \text{C.O.P.}$ then T_2 was changed to $T_2 - \Delta T_2$ and another C.O.P. was computed,

$$\text{C.O.P.}_2' = \text{C.O.P.}(T_2 - \Delta T_2,\ T_3, \ldots T_{n-1}) \ .$$

If $\text{C.O.P.}_2' > \text{C.O.P.}$ then changing T_2 to $T_2 - \Delta T_2$ was apparently in the right direction and the program proceeded to change T_3 to $T_3 + \Delta T_3$. However, if $\text{C.O.P.}_2' < \text{C.O.P.}$ then any change in T_2 tended to lower the C.O.P. and so the value of T_2 remained unchanged, and the program proceeded to change T_3. This procedure was repeated for each interstage temperature through T_{n-1}. If at this point, after varying all of the interstage temperatures, a net gain had occurred over the initial C.O.P., the program returned to vary T_2 and repeat the procedure.

APPENDIX 3

DEVELOPMENT OF A SIMPLIFIED APPROXIMATION FOR THE RATIO OF CASCADE C.O.P. TO SINGLE-STAGE C.O.P.

For any single stage thermocouple the maximum attainable C.O.P. is given by equation (16),

$$C.O.P._{max} = \frac{T_m(\sqrt{1 + ZT_m} - 1)}{\Delta T(\sqrt{1 + ZT_m} + 1)} - \frac{1}{2} . \qquad (3\text{-}1)$$

For the sake of simplicity define

$$M = \sqrt{1 + ZT_m} . \qquad (3\text{-}2)$$

Now the C.O.P. can be expressed

$$C.O.P._{max} = \frac{T_m}{\Delta T}\frac{M - 1}{M + 1} - \frac{1}{2} = \frac{2T_m(M-1) - \Delta T(M+1)}{2\Delta T(M + 1)} \qquad (3\text{-}3)$$

And since the maximum value of ΔT occurs at $C.O.P._{max} = 0$,

$$\Delta T_{max} = 2T_m \frac{M - 1}{M + 1} . \qquad (3\text{-}4)$$

Next for a cascade system the overall C.O.P. is given by equation (20)

$$1 + \frac{1}{C.O.P.} = \prod_{i=1}^{n} (1 + \frac{1}{C.O.P._i}) \qquad (3\text{-}5)$$

which can be rewritten using equations (3-1) and (3-2) ,

$$1 + \frac{1}{\text{C.O.P.}} = \prod_{i=1}^{n} \frac{2T_{m_i}(M_i - 1) + \Delta T_i (M_i + 1)}{2T_{m_i}(M_i - 1) - \Delta T_i (M_i + 1)} . \tag{3-6}$$

At this point digress momentarily to study the behavior of

$$M_i = \sqrt{1 + ZT_{m_i}} .$$

In general, $Z < 4 \times 10^{-3} \frac{1}{^\circ K}$ and also the variation of the mean stage temperature will be less than 100°K. So in order to estimate the greatest probable variation in M_i consider the following two values.

(a)

$$Z = 4 \times 10^{-3} \frac{1}{^\circ K}$$

$$T_{m_1} = 200^\circ K$$

$$ZT_{m_1} = 0.8$$

$$M_1 = \sqrt{1.8} = 1.342$$

(b)

$$Z = 3 \times 10^{-3} \frac{1}{^\circ K}$$

$$T_{m_2} = 300^\circ K$$

$$ZT_{m_2} = 1.2$$

$$M_2 = \sqrt{2.2} = 1.483$$

Thus the maximum percentage variation is

$$\frac{\Delta M}{M_1} \times 100 = \frac{0.141}{1.342} \times 100 = 10.51\% \text{ .}$$

Now assume that $M_i = M = \text{constant}$, where

$$M = \sqrt{1 + Z \frac{T_c + T_h}{2}} \text{ .} \tag{3-7}$$

For the example cited above

$$M = \sqrt{1 + 1} = 1.414 \text{ ,}$$

and the maximum percentage variation is now

$$\frac{\Delta M}{M} \times 100 = 4.92\%$$

Therefore it is possible to conclude that even in extreme cases the error involved in assuming M_i is a constant will be less than five per cent. The error introduced in the final approximation will be checked later.

Assuming that M_i is a constant, equation (3-6) becomes

$$1 + \frac{1}{\text{C.O.P.}} = \prod_{i=1}^{n} \frac{2T_{m_i}(M-1) + \Delta T_i(M+1)}{2T_{m_i}(M-1) - \Delta T_i(M+1)} \text{ .} \tag{3-8}$$

Now at optimal C.O.P. it has been shown by equation (24) that

$$\frac{T_i}{T_{i-1}} = \frac{T_{i+1}}{T_i} \text{ .}$$

Rewriting yields,

$$\frac{T_{i-1} + \Delta T_{i-1}}{T_i} = \frac{T_i + \Delta T_i}{T_i} .$$

or simplifying,

$$\Delta T_i = \frac{T_i}{T_{i-1}} \Delta T_{i-1} .$$

Since the temperatures are absolute, at normal working conditions $\frac{T_i}{T_{i-1}}$ is not much different from unity. From Figure 4 it can be concluded that a slight variation of the interstage temperature from optimal conditions will scarcely affect the C.O.P. Thus it introduces very little error to conclude that

$$\Delta T_i \cong \Delta T_2 \cong \Delta T_3 \cong \ldots \Delta T_n . \tag{3-9}$$

And, in general, for n stages,

$$\Delta T_i = \frac{\Delta T}{n}$$

where

$$\Delta T = T_h - T_c .$$

also

$$T_{m_i} = [(i-1) + \frac{1}{2}]\Delta T + T_c ,$$

Next substitute (3-10) and (3-11) into (3-8) and obtain,

$$1 + \frac{1}{C.O.P.} = \prod_{i=1}^{n} \frac{(2i-1)\Delta T(M-1) + \frac{\Delta T}{h}(M+1) + 2T_c(M-1)}{(2i-1)\Delta T(M-1) + \frac{\Delta T}{h}(M+1) + 2T_c(M-1)} .$$

Divide through by (M-1),

$$1 + \frac{1}{\text{C.O.P.}} = \prod_{i=1}^{n} \frac{(2i-1)\Delta T + 2T_c + \frac{\Delta T}{n}\frac{M+1}{M-1}}{(2i-1)\Delta T + 2T_c + \frac{\Delta T}{n}\frac{M+1}{M-1}} . \tag{3-12}$$

Next from equations (3-4) and (3-11)

$$\Delta T_{i_{max}} = [(2i-1)\Delta T + 2T_c]\frac{M-1}{M+1} ,$$

upon rearranging

$$(2i-1)\Delta T + 2T_c = \frac{M+1}{M-1}\Delta T_{i_{max}} .$$

Next substitute the above relationship into (3-12) and obtain

$$1 + \frac{1}{\text{C.O.P.}} = \prod_{i=1}^{n} \frac{\Delta T_{i_{max}} + \frac{\Delta T}{n}}{\Delta T_{i_{max}} - \frac{\Delta T}{n}} - 1 . \tag{3-13}$$

Now for a single-stage thermocouple operating between the temperatures T_h and T_c, the C.O.P. can be expressed from equations (3-3) and (3-4) as follows:

$$\text{C.O.P.}_{\text{single stage}} = \frac{\Delta T_{max} - \Delta T}{2\Delta T} . \tag{3-14}$$

Combining equations (3-13) and (3-14) after rearranging yields,

$$\frac{\text{C.O.P.}_{\text{cascade}}}{\text{C.O.P.}_{\text{single stage}}} = \frac{2\Delta T}{\Delta T_{max} - \Delta T} \times \frac{\prod_{i=1}^{n}\left(\Delta T_{i_{max}} - \frac{\Delta T}{n}\right)}{\prod_{i=1}^{n}\left(\Delta T_{i_{max}} + \frac{\Delta T}{n}\right) - \prod_{i=1}^{n}\left(\Delta T_{i_{max}} - \frac{\Delta T}{n}\right)} . \tag{3.15}$$

Now it is easily seen that

$$\Delta T_{i_{max}} = \frac{T_{m_i}}{T_m} \Delta T_{max}$$

Thus as a final assumption let

$$\Delta T_{i_{max}} = \Delta T_{max} .$$

With this assumption the repeated product can be expressed as a power and finally

$$\frac{C.O.P._{cascade}}{C.O.P._{single\ stage}} = \frac{2\,\Delta T}{\Delta T_{max} - \Delta T} \; \frac{(\Delta T_{max} - \frac{\Delta T}{n})^n}{(\Delta T_{max} + \frac{\Delta T}{n})^n - (\Delta T_{max} - \frac{\Delta T}{n})^n} . \quad (3\text{-}16)$$

The results of the approximate equation are compared with actual results in Table 9.

From the table it is apparent that the approximate equations yield good results. So equation (3-16) gives a good idea of the improvement gained by cascading without the great deal of mathematical computations involved in obtaining the exact value.

Table 9. Results of Approximate Cascade Equation

Stages	Actual Ratio	Approximated Ratio	Percent Error
$Z = 0.002 \frac{1}{^\circ K}$			
2	1.97	2.00	1.52
3	2.12	2.12	0.00
4	2.18	2.21	1.37
$Z = 0.003 \frac{1}{^\circ K}$			
2	1.24	1.23	0.41
3	1.28	1.28	0.47
4	1.29	1.29	0.39
$Z = 0.004 \frac{1}{^\circ K}$			
2	1.11	1.12	0.99
3	1.13	1.14	0.62
4	1.14	1.16	2.02
$Z = 0.005 \frac{1}{^\circ K}$			
2	1.07	1.08	0.56
3	1.09	1.10	1.19
4	1.19	1.10	2.20

APPENDIX 4

EFFECTS OF INCREASING THE HEAT REMOVAL ABOVE NORMAL FOR A TWO-STAGE CASCADE

Consider a two stage cascade for which Q_c is the designed cooling rate of the first stage (based on optimal C.O.P.) and φ_1 is the optimal value of the C.O.P. of the first stage. The heat which the second stage normally removes can be expressed as

$$Q_2 = Q_c + W_1 = Q_c + \frac{Q_c}{\varphi_1} = Q_c \left[1 + \frac{1}{\varphi_1}\right].$$

Suppose a change in the heat load occurs at the cold junction and Q_c increases to

$$Q_c' = \lambda\, Q_c$$

and simultaneously φ_1, changes to $\beta\varphi_1$. This necessitates an increase in the cooling rate of the second stage given by

$$Q_2' = \lambda Q_c \left[1 + \frac{1}{\beta\varphi_1}\right].$$

Thus the load on the first stage is increased such that

$$\frac{Q_c'}{Q_c} = \lambda\ .$$

At the same time the load on the second stage is increased to

$$\frac{Q_2'}{Q_2} = \frac{\lambda\left[1 + \frac{1}{\beta\varphi_1}\right]}{\left[1 + \frac{1}{\varphi_1}\right]} = \frac{\beta\varphi_1 + 1}{\beta\varphi_1 + \beta}\lambda \ .$$

Now since $\lambda > 1$ and $\beta < 1$ (since any change from optimal C.O.P. must decrease φ_1), the load increase is proportionately more than that on the first stage. Thus the maximum rate of cooling of a two stage device is limited by the capacity of the second stage.

APPENDIX 5

GOVERNING EQUATION FOR THE MODIFIED CASCADE

Consider the modified cascade below with its electrical circuit superimposed upon it:

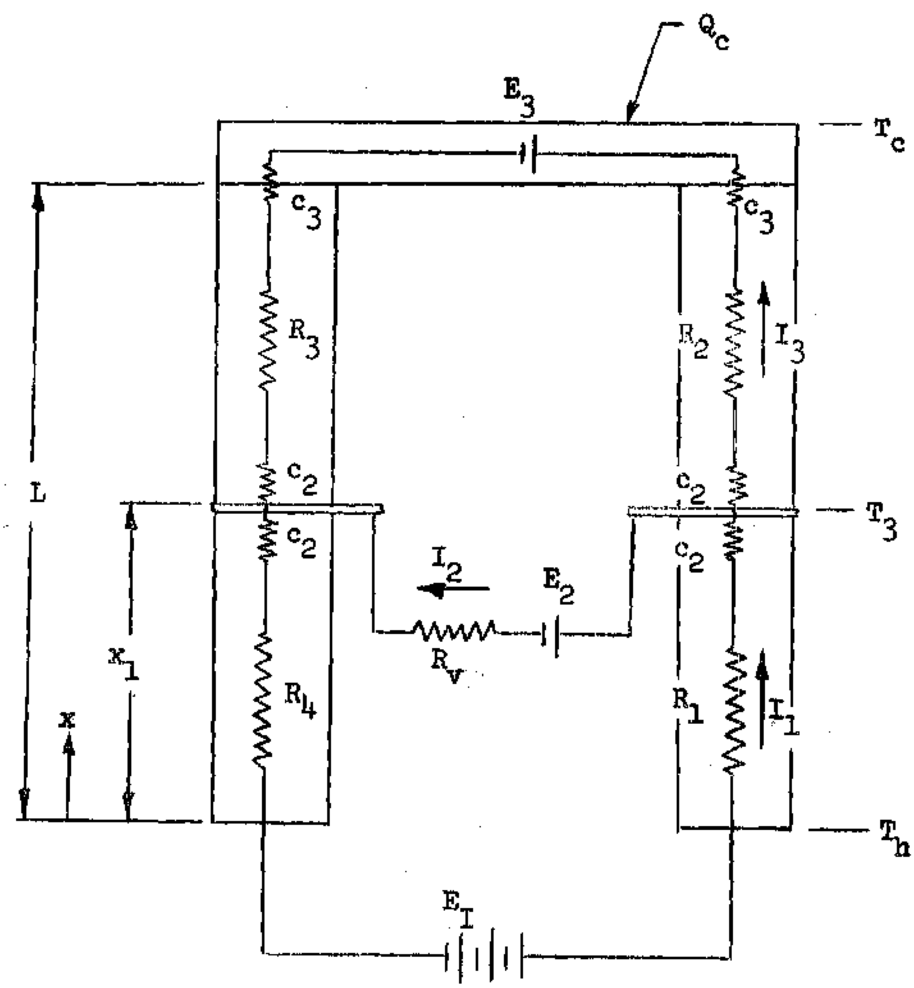

Figure 20. Schematic Representation of the Modified Cascade Thermocouple.

In the figure, R_1, R_2, R_3, and R_4 are the resistances of the legs of the semiconductor. Furthermore,

$$R_1 = R_4 = \frac{\rho x_1}{A}$$

and

$$R_2 = R_3 = \frac{\rho (L - x_1)}{A}.$$

Next c_2 and c_3 are the electrical contact resistances at soldered junctions. E_2 and E_3 are the back emf's resulting from the Seebeck effect.

$$E_2 = \alpha_{np}(T_h - T_3)$$

$$E_3 = \alpha_{np}(T_3 - T_c)$$

R_v is a resistor inserted in the circuit to control the center tap current, I_2.

The Ideal Situation

If contact resistances are neglected, then from circuit considerations the following equations are found:

$$I_2 = I_1 - I_3 \tag{5-1}$$

$$E_I = 2I_1R_1 + I_2R_v + \alpha_{np}(T_h - T_3) \tag{5-2}$$

$$I_2R_v + \alpha_{np}(T_h + T_c - 2T_3) - 2I_3R_2 = 0 \tag{5-3}$$

From the above equations R_v and E_I are then found. Next perform an energy balance at the center tap. If steady-state conditions prevail, T_3 = constant and thus,

$$Q_{C.T._{net}} = 0 \; .$$

Summing energies at the center tap,

$$Q_{C.T._{net}} = 0 = \frac{1}{2} I_3^2 (2R_2) + \frac{1}{2} I_1^2 (2R_1) - I_2 \alpha_{np} T_3$$

$$- 2K_2(T_3 - T_c) + 2K_1(T_h - T_3) \; .$$

Rearranging yields,

$$T_3 = \frac{I_3^2 R_2 + I_1^2 R_1 + 2K_2 T_c + 2K_1 T_h}{\alpha_{np} I_2 + 2K_1 + 2K_2} \qquad (5\text{-}4)$$

where

$$K_1 = \frac{A K}{x_1}$$

$$K_2 = \frac{A K}{L - x_1} \; .$$

Thus it is seen that T_3 is a function of I_1, I_3 and x_1. The heat removal at the cold junction is given by

$$Q_c = I_3 \alpha_{np} T_c - 2K_2(T_3 - T_c) - I_3^2 R_2 \; . \qquad (5\text{-}5)$$

The energy required to remove the heat, Q_c, is expressed,

$$W = I_1^{\,2}(2R_1) + I_3(2R_2) + I_2^{\,2}R_v + \alpha_{np}(T_h - T_c)I_3 + \alpha_{np}(T_h - T_3)I_2$$

where the first three terms on the right side of the equation result from Joule heating and the remaining terms are due to overcoming the Seebeck voltage. Now by definition the C.O.P. is written

$$\text{C.O.P.} = \frac{Q_c}{W} \ .$$

When the expressions for Q_c, W, T_3, R_1, R_2, K_1 and K_2 are substituted into the above equation, the result is equation (26)

Contact Resistance Considerations

The same procedure is used as in developing the equations without the contact resistance. The first step is to perform an energy balance at the center tap. Again steady-state conditions prevail and thus,

$$0 = \frac{1}{2}\, I_3^{\,2}(2R_2) + \frac{1}{2}\, I_1^{\,2}(2R_1) + I_1^{\,2}(2c_2) + I_3^{\,2}(2c_2) - I_2\alpha_{np}T_3 - 2K_2(T_3 - T_c) + 2K_1(T_h - T_3) \ .$$

Rearranging yields,

$$T_3 = \frac{I_3^{\,2}(R_2 + 2c_2) + I_1^{\,2}(R_1 + 2c_2) + 2K_2T_c + 2K_1T_h}{\alpha_{np}I_2 + 2K_2 + 2K_1} \ . \qquad (5\text{-}7)$$

Next for the heat removal at the cold junction,

$$Q_c = I_3\alpha_{np}T_c - 2K_2(T_3 - T_c) - I_3^{\,2}(R_2 + 2c_3) \ . \qquad (5\text{-}8)$$

The input energy is expressed:

$$W = I_1^2(2R_1) + I_3^2(2R_2) + I_1^2(2c_2) + I_3^2(2c_2 + 2c_3) + I_2^2 R_v$$

$$+ \alpha_{np}(T_h - T_c)I_3 + \alpha_{np}(T_h - T_3)I_2 \,. \qquad (5\text{-}9)$$

The same expressions hold for R_1, R_2, K_1 and K_2 as in the case for no contact resistance. Equation (5-7) is solved for T_3, and then equations (5-8) and (5-9) are solved for Q_c and W. Finally the C.O.P. is given by $\frac{Q_c}{W}$.

APPENDIX 6

HEAT TRANSFER ANALYSIS OF THERMOCOUPLES

For the leg of a thermocouple, the one-dimensional heat conduction equation will apply if heat transfer from the sides of the leg is neglected. Furthermore, if steady-state conditions prevail, and Joule heating is treated as an internal homogeneous heat source, the governing equation can be written,

$$k \frac{d^2T}{dx^2} + Q''' = 0 \qquad (6\text{-}1)$$

where

$$Q''' = I^2 \frac{\rho}{A^2} .$$

Assume that as boundary conditions the temperatures at both ends of the leg are known. That is, at $x = 0$, $T = T_h$ and at $x = L$, $T = T_c$. The solution to equation (6-1) now becomes

$$T = \frac{1}{2} \frac{I^2 \rho x}{kA^2} (L - x) + T_h - (T_h - T_c) \frac{x}{L} . \qquad (6\text{-}2)$$

Applying this to the modified thermocouple with a center tap at $x_1 = 0.5$ cm and

$$L = 1.0 \text{ cm}$$

$$A = 0.370 \text{ cm}$$

$$T_c = 250°K$$

$$T_h = 300°K$$

$$\rho = 10^{-3} \text{ ohm cm}$$

$$I_1 = 22 \text{ amps}$$

$$I_2 = 11 \text{ amps}$$

the equation becomes

$$T = 300 - 10.2 \frac{x}{0.5} - 22.1 \left(\frac{x}{0.5}\right)^2 \text{ for } 0 \leq x \leq 0.5, \qquad (6\text{-}3)$$

and for $x > x_1$ equation (5-2) is written,

$$T = 267.7 - 12.2 \frac{x}{0.5} - 5.53 \left(\frac{x}{0.5}\right)^2 \text{ for } 0.5 \leq x \leq 1.0. \qquad (6\text{-}4)$$

Next consider a simple thermocouple of the same size with a current of 22 amps. The temperature distribution is given by

$$T = 300 = 38.5 \frac{x}{1.0} - 88.5 \left(\frac{x}{1.0}\right)^2 . \qquad (6\text{-}5)$$

From Appendix 1 equations can be found which will give the optimal current for this thermocouple. In this case $I_{opt} = 14.41$ amps. With this current the temperature distribution becomes,

$$T = 300 - 11.9 \frac{x}{1.0} - 38.1 \left(\frac{x}{1.0}\right)^2 . \qquad (6\text{-}6)$$

Equations (6-3), (6-4), (6-5), and (6-6) are plotted in Figure 8.

Conduction Heating at Cold Junction

For the simple thermocouple

$$Q = \frac{2\kappa A}{L}\,(T_h - T_c) = \frac{2(0.02 \text{ watts})}{\text{cm } ^\circ\text{K}}\,\frac{0.370 \text{ cm}^2}{1 \text{ cm}}\,(50^\circ\text{K})$$

which reduces to

$$Q = 0.740 \text{ watts} .$$

For the modified thermocouple, solution of equation (6-4) yields $T_3 = 267.7^\circ$K. So that the conduction heating is expressed

$$Q = \frac{2\kappa A}{L - x_1}\,(T_3 - T_c) = 0.525 \text{ watts.}$$

Joule Heating at the Cold Junction

For the simple thermocouple three different currents are considered. In each case one half of the Joule heat goes to the cold junction and the total resistance is given by

$$R = \frac{2\rho L}{A} = 5.4 \times 10^{-3} \text{ ohms} .$$

(1) For $I = 22$ amps

$$Q = 1/2\ I^2R = 1.308 \text{ watts}$$

(2) For $I = 11$ amps

$$Q = 1/2\ I^2R = 0.337 \text{ watts}$$

(3) For $I = 14.41$ amps (optimal value)

$$Q = 1/2\ I^2R = 0.560 \text{ watts}$$

For the modified thermocouple one half of the Joule heating in the section of the couple for $x > 0.5$ cm is deposited at the cold junction. For this section the resistance is given by

$$R = \frac{2\rho(L - x_1)}{A} = 2.70 \times 10^{-3} \text{ ohms} .$$

Thus the Joule heat deposited at the cold junction is

$$Q = \frac{1}{2} I^2R = \frac{1}{2}\ (11 \text{ amps})^2(2.70 \times 10^{-3} \text{ ohms}) = 0.163 \text{ watts} .$$

<u>Heat Removal by Peltier Cooling</u>

Again for the simple thermocouple each of the three different currents must be considered.

(1) For $I = 22$ amps

$$Q = \alpha_{np}\ T_cI = 2.335 \text{ watts}$$

(2) For $I = 11$ amps

$$Q = \alpha_{np}\ T_cI = 1.167 \text{ watts}$$

(3) For $I = 14.14$ amps

$$Q = \alpha_{np}\ T_cI = 1.528 \text{ watts}$$

For the modified thermocouple, the cold junction current is used and

$$Q = \alpha_{np}\ T_cI = 1.167 \text{ watts} .$$

The work input to the simple thermocouples is given by the equation

$$W = 2I^2R + \alpha_{np}\, I \Delta T \, .$$

For the modified thermocouple the input work is given by

$$W = 2I_1^{\,2}R_1 + 2I_3^{\,2}R_2 + I_2^{\,2}R_v + \alpha_{np}I_2(T_h - T_3) + \alpha_{np}I_3(T_h - T_c) \, .$$

All of the values are tabulated for study in Table 5.

APPENDIX 7

THE DEVELOPMENT OF AN EXPRESSION TO OBTAIN THE MAXIMUM TEMPERATURE DIFFERENCE FOR THE MODIFIED CASCADE

The Ideal Situation

As the thermocouple approaches its limiting value for ΔT_{max}, the amount of heat removed from the cold junction approaches zero. From equations (5-4) and (5-5), with Q_c equal to zero, the following may be obtained,

$$0 = I_3 \alpha_{np} T_c + 2K_2 T_c - I_3^2 R_2 - 2K_2 \frac{I_3^2 R_2 + I_1^2 R_1 + 2K_2 T_c + 2K_1 T_h}{\alpha_{np} I_2 + 2K_1 + 2K_2} .$$

After rearranging this expression becomes

$$T_h = \frac{(\alpha_{np} I_2 + 2K_1 + 2K_2)(I_3 \alpha_{np} T_c + 2K_2 T_c - I_3^2 R_2)}{4K_1 K_2} - \frac{I_3^2 R_2 + I_1^2 R_1 + 2K_2 T_c}{2K_1} , \qquad (7\text{-}1)$$

where

$$K_1 = \frac{2A\kappa}{x_1} , \qquad K_2 = \frac{2A\kappa}{L - x_1} ,$$

$$R_1 = \frac{\rho x_1}{A} , \qquad R_2 = \frac{\rho (L - x_1)}{A} ,$$

and $I_2 = I_1 - I_3$.

Now if it is assumed that the material is already determined, then T_h becomes a function of I_1, I_3, L, x_1, A and T_c. Since an analytic solution to equation (7-1) would be very difficult, a method of numerical optimization was used on a digital computer.

Contact Resistance Considerations

As ΔT approaches ΔT_{max}, Q_c approaches zero. Thus from equations (5-7) and (5-8) can be written

$$0 = I_3\alpha_{np}T_c + 2K_2T_c - I_3^2(R_2 + 2c_3)$$

$$- 2K_2 \frac{I_3^2(R_2 + 2c_2) + I_1^2(R_1 + 2c_2) + 2K_2T_c + 2K_1T_h}{\alpha_{np}I_2 + 2K_2 + 2K_1}$$

Next the equation above is solved for T_c.

$$T_c\left[I_3\alpha_{np} + 2K_2 - \frac{4K_2^2}{\alpha_{np}I_2 + 2(K_2 + K_1)}\right] =$$

$$I_3^2(R_2 + 2c_3) + 2K_2 \frac{I_3^2(R_2 + 2c_2) + I_1^2(R_1 + 2c_2) + 2K_1T_h}{\alpha_{np}I_2 + 2(K_1 + K_2)} \qquad (7\text{-}2)$$

Equation (7-2) was solved for T_c for $T_h = 300°K$ for the extremal values of the properties given in Appendix 9. These determined the limiting curves plotted in Figure 14.

APPENDIX 8

PROPERTY VALUES OF MELCOR

The property values of the thermoelements were determined at a temperature of 300°K, and are the courtesy of Allen D. Reich, Borg-Warner Corporation, Des Plains, Illinois. These properties are listed below

	'n' Type Element	'p' Type Element
Electrical Resistivity (ohm-cm)	0.000860 to 0.000950	0.000860 to 0.000950
Seebeck Coefficient (volts/°K)	0.000198 to 0.000210	0.000190 to 0.000195
Thermal Conductivity	0.0137 to 0.0167	0.0140 to 0.0145
Figure of Merit*	0.00270 to 0.00302	0.00260 to 0.00290

*The figure of merit values were obtained from measurements of the maximum temperature difference.

APPENDIX 9

PROPERTIES OF SANTOCEL 'A' INSULATION

Santocel 'A' is a silica aerogel manufactured by Monsanto chemical, Inorganic Chemicals Division, St. Louis, 66, Missouri. Because of its unusually low thermal conductivity it was used to insulate the thermo-elements. It should be noted that its thermal conductivity is somewhat lower than the thermal conductivity of air at the same temperature. Some of the properties of this insulation are given below.

Typical Physical Properties

Form	Fluffy, white powder and granules
Color	White
Loose Bulk Density	5.0 lb/ft
Absolute Density	2.2 gm/cc
Pore Volume	93 - 95%
Specific Heat	0.234 cal/gm/°C
Temperature Range	Liquid Helium to 1300°F
Screen Size	+8 Mesh 1.0% max
	-40 Mesh 35.0% max

Thermal Conductivity

Temperature (°F)	K $\left(\frac{\text{Btu in}}{\text{hr ft}^2\ ^\circ\text{F}}\right)$
-100	0.117
0	0.138
100	0.160
200	0.184
300	0.212

APPENDIX 10

DETERMINATION OF CONTACT RESISTANCES

In order to describe the process of determining contact resistances the schematic representation of the element in Figure 21 is used.

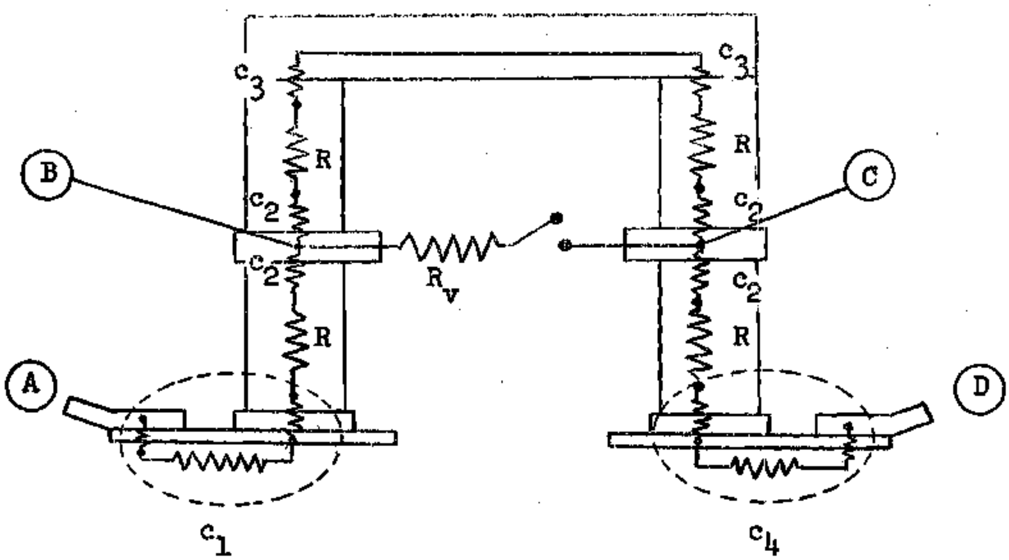

Figure 21. Schematic of the Modified Cascade Thermocouple.

Since it is very difficult to accurately measure small resistances, a current of known value was passed from Point (A) to Point (D) and the voltage was measured between Points (A), (B), and (C), and (D), with a potentiometer. It was then possible to find the resistances from Ohm's law. Thus the current and V_{AB}, V_{BC}, and V_{CD} are known. Since it is physically impossible to measure any other voltage drops around the circuit, some approximations must be made. From the property data on the semiconductors the value of the resistance of the element legs was assumed to be known. Next since the four contact resistances at the center tap were physically similar, they were assumed to be equal and designated as C_2. Finally the two contact resistances at the cold junctions were assumed to be equal and designated as C_3. In order to eliminate a back emf due to the Seebeck effect, the cold junction was heated (in order to compensate for the Peltier cooling), so that it was at the same temperature as the hot junction. The following data was obtained:

$T_h = 290°K$ Potentiometer Junction

$T_3 = 290°K$ Temperature = 290°K

$T_c = 290°K$

$V_{AB} = 5.22$ mv $I = 1.482$ amps

$V_{BC} = 6.89$ mv

$V_{CD} = 6.08$ mv

The following equations can be written:

$$I(C_1 + R + C_2) = 5.22 \text{ mv}$$

$$I(2C_2 + 2R + 2C_3) = 6.89 \text{ mv}$$

$$I(C_4 + R + C_2) = 6.08 \text{ mv}$$

In order to solve these equations, it is necessary to assume that C_2 is equal to C_3. Since the junctions are physically the same this is not a bad assumption. From the property data the resistivity lies between 0.950×10^{-3} ohm-cm and 0.860×10^{-3} ohm-cm. Using these values, the resistance R lies between 1.90×10^{-3} ohms and 1.72×10^{-3} ohms. Using these two limits for R two extreme values can now be found for each of the contact resistances from the circuit equations. The values found are

$$0.228 \times 10^{-3} \leq C_2 = C_3 \leq 0.306 \times 10^{-3} \text{ ohms}$$

$$1.32 \times 10^{-3} \leq C_1 \leq 1.58 \times 10^{-3} \text{ ohms}$$

$$1.90 \times 10^{-3} \leq C_4 \leq 3.16 \times 10^{-3} \text{ ohms}$$

At this point some doubt arises as to the assumptions that the contact resistances are the same at cold junction and both sides of the center tap. In order to approach the problem in a different light consider an energy balance at the center tap when the couple is used as a simple thermocouple. If steady state conditions prevail the net energy passing through the junction is zero, and expressed mathematically it is stated:

$$0 = \frac{1}{2} I^2R_1 + \frac{1}{2} I^2R_2 + I^2(2C_2) + K_1(T_h - T_3) - K_2(T_3 - T_c) .$$

Since $R_1 = R_2$ and $K_1 = K_2$, simplification and rearrangement yields:

$$C_2 = \frac{K_1}{2I^2} (2T_3 - T_h - T_c) - \frac{1}{2} R .$$

In order to be sure that the heat leak is held to a very small amount, experimental values were obtained with the junction temperatures close to the ambient value of 297°K. The measured values for $I = 2.96$ amps were:

$$T_h = 297^\circ K$$

$$T_3 = 295^\circ K$$

$$T_c = 290^\circ K$$

Thus the equation for C_2 becomes:

$$C_2 = 0.171\ K_1 - \frac{1}{2} R_1 .$$

Using the property values given for ρ and κ in the preceding equation yields:

$$0.250 \times 10^{-3} \text{ ohms} \leq C_2 \leq 0.422 \times 10^{-3} \text{ ohms}$$

This is within the range found by the electrical circuit analysis.

Finally, a thermocouple was inserted at each junction of the center tap to measure the temperature. The temperatures at these two junctions were within a degree of each other for all currents up to twenty amps, and then they varied slightly. Thus the junction resistances must have been very nearly the same, as was assumed.

APPENDIX 11

EXPERIMENTAL DATA

The experimental data are presented in Tables 10, 11, 12 and 13. A word of explanation is needed to explain several of the entrees in the tables.

In calculation of the input power, the resistance of the leads from the power supply to the hot junction was not considered in the analytical development. Thus, in order to obtain a proper comparison between experimental and analytical results, the voltage drop across the thermocouple was corrected to eliminate the voltage drop through the leads. The corrected input voltage was given by the following expression:

$$E_c = E_I - I_1 \times R_{lead} .$$

The average value for R_{lead} was found to be 3.9×10^{-3} ohms in Appendix 10 and was used in all calculations.

The nichrome wire forming the cold junction heater had a resistance of 4.0 ohms. Thus the cooling capacity of the thermocouple was given by $4\ I_H^2$.

For the modified cascade the heat leak through the center tap leads needed to be estimated. The leads were number 14 (0.0409 in. diameter) solid copper wire with rubber insulation and approximately four inches long. The conductivity for pure copper at room temperature is 9.67 watts/°C in. For these leads, the conductivity is $K = 30.4 \times 10^{-4} \frac{\text{watts}}{°K\ \text{in}}$

for each lead so that the heat leak into the thermocouple is given by

$$6.08 \times 10^{-3}\ (T_{con} - T_3)\ \text{watts}\ .$$

Since the center tap is midway between the hot and cold junctions, one half of this heat is deposited at each junction of thermocouple. Therefore Q_c' can be expressed;

$$Q_c' = Q_c + 3.04 \times 10^{-3}(T_{con} - T_3)\ .$$

The estimation of the effect of the heat leak on the maximum temperature difference is more difficult. Assume that as an approximation the center tap temperature would not be changed if the leads were insulated. An energy balance at the cold junction yields;

$$Q_c = 0 = I_3\alpha_{np}T_c' - 2K_2(T_3 - T_c') - I_3^2(R_2 + 2C_3) + 3.04 \times 10^{-3}(T_{con} - T_3)\ . \quad (11\text{-}1)$$

In the preceding equation $T_c' = T_c + \Delta T_c$, where T_c is the cold junction temperature with no heat leak. Now if there were no heat leak at the cold junction an energy balance could be written,

$$Q_c = 0 = I_3\alpha_{np}T_c - 2K_2(T_3 - T_c) - I_3^2(R_2 + 2C_3)\ . \quad (11\text{-}2)$$

Subtracting equation (11-12) from (11-1) gives the following expression:

$$\Delta T_c = \frac{3.04 \times 10^{-3}(T_{con} - T_3)}{I_3\alpha_{np} + 2K_2}\ . \quad (11\text{-}3)$$

Suppose that the average property values are assumed for the Seebeck coefficient and thermal conductivity. These values substituted into equation (11-3) give the following equation for ΔT_c:

$$\Delta T_c = \frac{3.04\ (T_{con} - T_3)}{0.395\ I_3 + 14.5}$$

Now the corrected maximum temperature difference is given as the sum of the measured maximum temperature difference and ΔT_c.

The analytical curves for the C.O.P., heat removal and maximum temperature differences were obtained from solving the governing equations developed in Appendices 5 and 7.

Table 10

Simple Thermocouple (No Load)

Input Current (amps)	Hot Junction Temperature (°K)	Center Tap Temperature (°K)	Cold Junction Temperature (°K)	Maximum ΔT (°K)
6.030	299	284	251	48
7.860	299	285	243	56
9.084	299	282	240	59
10.926	300	287	244	56
13.932	301	293	247	54
15.210	301	312	251	50
16.974	302	311	252	50

Table 11. Simple Thermocouple (With Load)

Hot Junction Temperature = 300°K
Cold Junction Temperature = 250°K

Input Current (AMPS)	Input Voltage (Volts)	Corrected Input Voltage (Volts)	Input Power (Watts)	Heater Current (AMPS)	Cooling Capacity (Watts)	C.O.P.	T_3 (°K)
6.993	0.080	0.051	0.357	0.085	0.0289	0.081	288
8.130	0.115	0.083	0.675	0.135	0.0729	0.108	291
9.144	0.125	0.089	0.813	0.150	0.0900	0.110	295
10.290	0.135	0.095	0.978	0.160	0.1024	0.105	297
12.078	0.150	0.103	1.242	0.155	0.0960	0.077	303
13.110	0.165	0.114	1.493	0.135	0.0729	0.049	308

Table 12. Modified Cascade Thermocouple (No Load)

Hot Junction Current (Amps)	Cold Junction Current (Amps)	Hot Junction Temperature (°K)	Center Tap Temperature (°K)	Cold Junction Temperature (°K)	Maximum ΔT (°K)	C.R. Connector Temperature (°K)	Corrected Maximum ΔT (°K)
12.72	6.00	299	268	242	57	282	60.3
14.60	6.798	300	266	235	65	286	68.5
17.05	7.926	300	267	232	68	288	71.6
19.53	9.090	300	269	230	70	289	73.4
22.05	10.260	300	273	230	70	289	73.2

Table 13. Modified Cascade Thermocouple with Load

Hot Junction Temperature = 300°K
Cold Junction Temperature = 200°K

I_1 (amps)	I_3 (amps)	E_T (volts)	E_c (volts)	W (watts)	I_H (amps)	Q_c (watts)	C.O.P.	T_3 (°K)	T_{con} (°K)	Q_c' (watts)	C.O.P.'
10.25	4.79	0.105	0.065	0.666	0.155	0.096	0.144	271	288	0.147	0.221
12.72	5.94	0.130	0.081	1.031	0.205	0.168	0.163	271	287	0.216	0.209
14.60	6.80	0.157	0.100	1.460	0.245	0.240	0.165	272	288	0.288	0.197
17.05	7.93	0.190	0.124	2.115	0.265	0.281	0.133	273	288	0.326	0.154
19.53	9.09	0.202	0.122	2.382	0.280	0.314	0.132	277	289	0.350	0.147
22.05	10.26	0.255	0.169	3.730	0.280	0.314	0.084	281	289	0.335	0.090

BIBLIOGRAPHY

1. H. J. Goldsmid, Applications of Thermoelectricity, Methuen and Co., Ltd., London, 1960

2. C. N. Rollinger, "Convectively Cooled Thermoelements with Variable Cross-Sectional Area," Trans. American Society of Mechanical Engineers Journal, Vol. 87, 1965, pp. 259-265.

3. L. J. Ybarrondo and J. E. Sunderland, "Optimized Performance of a Thermoelectric Heat Pump with Surface Heat Transfer and Finite Fins," Advanced Energy Conversion, Vol. 4, pp. 71-84, Pergamon Press, 1964.

4. Arthur R. Foster, "Two Stage Thermoelectric Refrigeration," Trans. ASHRAE, Vol. 70, pp. 312-318, 1964.

5. F. E. Jaumot, Jr., "Thermoelectric Effects," Proceedings of the IRE, Vol. 46, No. 3, 1958, pp. 538-554.

6. Ralph Crump, "Design-In Your Own Thermoelectric Cooling," Product Engineering, Vol. 34, No. 19, 1963, pp. 81-89.

7. B. J. O'Brien, C. S. Wallace, and K. Landecker, "Cascading of Peltier Couples for Thermoelectric Cooling," Journal of Applied Physics, Vol. 27, No. 7, 1956, pp. 820-823.

8. A. F. Ioffe, Semiconductor Elements and Thermoelectric Cooling, Infosearch Ltd, London, 1957.

9. E. S. Rittner, "On the Theory of the Peltier Heat Pump," Journal of Applied Physics, Vol. 30, No. 5, 1959, pp. 702-707.

10. W. H. Clingman, "Entropy Production and Optimum Device Design," Advanced Energy Conversion, Vol. I, Part II, 1961, pp. 61-81.

11. Direct Energy Conversion Literature Abstracts, Edited U. S. Naval Research Laboratory, Dec. 1964.

12. J. M. Borrego, "Approximate Analysis of the Operation of Thermoelectric Generators with Temperature Dependent Parameters," IEEE Transactions, Feb. 1964.

13. R. L. Eichorn, "Thermoelectric Refrigeration," Refrigerating Engineering, Vol. 66, No. 6, 1958, pp. 31-35.

14. P. H. Egli, "Symposium on Thermoelectric Energy Conversion," Advanced Energy Conversion, Jan-June 1962, Pergamon Press Ltd., N. Y.

15. P. O. Gehloff et al., "Thermoelectric Refrigeration," Refrigerating Engineering, Vol. 58, 1950, p. 1079.

16. H. J. Goldsmid and R. W. Douglas, "The Use of Semi-Conductors in Thermoelectric Refrigeration," British Journal of Applied Physics, Vol. 5, No. 11, Nov. 1954, pp. 386-390.

17. H. J. Goldsmid, "Thermoelectric Applications of Semi-Conductors," Journal of Electronics, Vol. 1, No. 2, Sept. 1955, pp. 219-222.

18. J. Kronshein and W. O. Hartsaw, "Peltier Effect," Refrigerating Engineering, Vol. 66, No. 9, Sept. 1958, pp. 31-33.

19. B. J. O'Brien and C. S. Wallace, "Ettinghausen Effect and Thermomagnetic Cooling," Journal of Applied Physics, Vol. 29, No. 7, July 1958, pp. 1010-1012.

20. G. Rezek, "Thermal Design and Analog Representation of a Thermoelectric Refrigerator," IEEE Transactions on Product Engineering and Production, Sept. 1963, pp. 51.

21. I. I. Sochard, "Thermoelectric Cooling," Paper No. 1043-59, American Rocket Society, 14th Annual Meeting, Nov. 1959.

22. S. Uchiyama, "Theory of Thermoelectric Refrigeration," Memoirs of the Faculty of Engineering Nagota University, Vol. 9, No. 2, Nov. 1957, pp. 321-329.

23. B. Varga, A. D. Reich and J. R. Madigan, "Thermoelectric and Magnetic Heat Pumps," Journal of Applied Physics, Dec. 1963, p. 3430.

24. W. B. Green, Editor, "Westinghouse Thermoelectric Handbook," Westinghouse Electric Corp., Semi-Conductor Division, Youngwood, Penn., 1962.

25. T. C. Harmon, "Multiple Stage Thermoelectric Generation of Power," Journal of Applied Physics, Vol. 29, No. 10, Oct. 1958, pp. 1471-1473.

26. A. I. Burshtein, "Concerning Efficiency of Cascade Thermogenerators," translated from Fizika Tverdogo Tela, Vol. 2, No. 10, Oct. 1960, pp. 2505-2508.

27. S. W. Angrist and M. Vallidis, "An Experiment in Coupled Flows," Journal of Engineering Education, Vol. 55, No. 9, May 1965, pp. 257-265.

VITA

Peter W. Cowling was born in Neenah, Wisconsin on February 7, 1939. At the age of 8 he moved to Memphis, Tennessee. He was graduated from Christian Brothers High School in Memphis in 1957.

In September 1957 he entered the University of Tennessee, from which he received his B. S. degree in Mechanical Engineering in June 1962. During this period he worked on the Engineering Cooperative program with Kimberly-Clark Corporation in Memphis, Tennessee. In September of 1962 he entered Oklahoma State University on a teaching assistantship. From this institution he received a M. S. degree in Mechanical Engineering in June of 1964. Since that time he has been studying at Georgia Institute of Technology on a NASA fellowship.

Mr. Cowling was married in 1962 to the former Frieda Gay White. They have a daughter, Kelly, who is 3 months old.